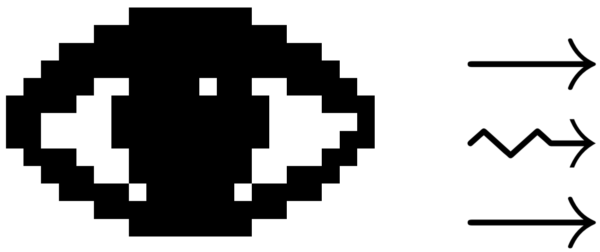
Course Notes

Mathematical Models, Spring 2010

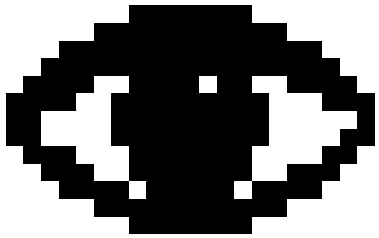
What is Mathematical Modeling?



observed real-world behavior

- What is behind the behavior?
- How can we measure what is happening?
- What data can we collect?
- How can we convey that our reasoning is plausible?

Steps of the Modeling Process



$\longrightarrow$ $\leadsto$ $\longrightarrow$ observed real-world behavior

First Step: Formulation

- *State the question.* If the question is vague, make it precise. If the question is too big, subdivide it into manageable parts.
- *Identify factors.* Decide which quantities influence the behavior. Determine relationships between the quantities.

  [*In this step, we are introducing* ________________________.]
- *Describe mathematically.* Assign each quantity a variable. Represent each relationship with an equation.

Motivating Example: Gravity by Galileo

In Galileo's time the question changed from:
Why do objects fall? (Philosophical question) to
How do objects fall? (How to describe a falling object's velocity?)

First Step: Formulation.

- *State the question.*
  What formula describes how an object gains velocity as it falls?
- *Identify factors.* Galileo chose only distance, time, and velocity.

  Other variables:________________________________

  Assumption: Velocity is proportional to the distance fallen.
- *Describe mathematically.*
  Assign variables to distance (x), time (t), and velocity (v).
  Relationships give equations: velocity and distance satisfy $v = \frac{dx}{dt}$.
  Proportionality implies $v = ax$ for some constant a.

Steps of the Modeling Process

After formulation we have some variables and equations, and now we have to do some sort of analysis of these equations in order to develop some sort of **mathematical conclusions**.

Second Step: Mathematical Manipulation.
This may entail one or more of:

- Calculations
- Proving a theorem
- Solving an equation
- Other...

Motivating Example: Gravity by Galileo
Since $v = \frac{dx}{dt}$ and $v = ax$, we set equal the two equations.

This gives the (differential) equation: $\frac{dx}{dt} = ax$.

Solving gives that $x(t) = ke^{at}$ for some constants a and k.

Something is not quite right...

Steps of the Modeling Process

We have a mathematical conclusion, but does it give a "right answer"?

Perhaps the most important, but least considered step of the modeling process is:

Third Step: Evaluation. Translate the mathematical results back to the real-world situation and ask the following questions:

- Has the model explained the real-world observations?
- Are the answers we found accurate enough?
- Were our assumptions good assumptions?
- What are the strengths and weaknesses of our model?
- Did we make any mistakes in our mathematical manipulations?

If there are any problems, we need to return to the formulation step and return through the modeling process.

Motivating Example: Gravity by Galileo

Third Step: Evaluation.

Through our mathematical calculations, we have determined that the position of a falling object is given by the function $x(t) = ke^{at}$.

The real-world situation we are modeling is starting from rest at time zero. That is, $x(0) = 0$. This implies that $0 = ke^{a0} = ke^{0} = k$, and therefore, $x(t) = 0$.

In words, this means that the object stays at rest for all t.

This is absurd; perhaps the proportionality assumption is incorrect.

Motivating Example: Gravity by Galileo

First Step: Formulation.

Q. What formula describes how an object gains velocity as it falls?

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

Second Step: Mathematical Manipulation.

Integrating gives $x(t) = 16t^2 + C$. Since $x(0) = 0$ we can find $C = 0$.

Therefore an object falling from rest has position $x(t) = 16t^2$.

Third Step: Evaluation.

This function agrees well with observations in many instances. (Although not all!)

The Modeling Process

This chart summarizes the modeling process.