MATH 636, Fall 2013
PRACTICE PROBLEMS
in preparation for Exam 2 on Tuesday, December 10, 2013

The exam covers:

- *Combinatorics: A Guided Tour*, Sections 2.4, 3.3–3.6 (not including EGFs), 4.1, 4.4.

- Additional topics that are included in the course notes, including generating function manipulations and applications, products and compositions of generating functions, compositions of an integer, standard Young tableaux, Catalan numbers, combinatorial statistics, and q-analogs.

Below are some questions that practice concepts from the class.

- Book exercises: 2.4.6, **2.4.8**, 3.3.5, 3.3.9 (also solve using convolution), 3.5.1e, **3.6.2**, 3.6.6, 4.1.13, 4.4.1, 4.4.3, **4.4.9**, (Bolded exercises are especially encouraged.)

- Book questions (in the chapters): 123, 172, 174

- *Challenge questions:* 3.4.11

Q1. We can make a four-sided die using a tetrahedron. If the sides of the die are 1, 2, 3, and 4, then there is a particular distribution of possible sums between 2 and 8. Determine a relabeling of the two dice which gives the exact same distribution of sums. How many possibilities are there?

Q2. The Catalan numbers c_n satisfy the generating function

$$C(x) = \sum_{n \geq 0} c_n x^n = \frac{1 - \sqrt{1 - 4x}}{2}$$

Consider an integer sequence $\{d_n\}_{n \geq 0}$ that has as its generating function

$$D(x) = \sum_{n \geq 0} d_n x^n = 1 - 5\sqrt{1 - 4x}.$$

Determine a formula for d_n written as a function of Catalan numbers.

Q3. Investigate and conjecture a formula for the number of standard Young tableaux of shape $\lambda : 2n = n + n$ for all positive n. For example, when $n = 3$, two valid tableaux

are
1	2	4
3	5	6
 and
1	3	5
2	4	6
. Prove your conjecture by finding a bijection with a combinatorial interpretation we have seen previously.

[*Hint: You will likely use the fact that a standard Young tableau of shape (n, n) is completely defined by its first row.*]

Q4. Show that the number of Young diagrams that fit inside an $m \times n$ rectangle is $\binom{m+n}{m}$. [*Hint: look for a bijection with walks in a lattice.*]