MATH 636, Fall 2013
Homework 8
To be prepared for presentation on Tuesday, December 3.

Background reading: Combinatorics: A Guided Tour, Section 4.1 along with the notes on Catalan numbers, combinatorial statistics, and q-analogs.

If you wish to present one of these questions in class, claim it upon arrival. (If you have already presented, please let others present this time.)

As with Homework 6, you have the option of turning in your homework for grading on a ✓+, ✓, ✓−, ✗ scale—your grade be computed by only considering your best *three* answers. Turn in your homework at the beginning of class.

8-1. Suppose you have an unlimited supply of black building blocks of height 1 and an unlimited supply of red, orange, yellow, green, blue, and purple building blocks of height 2. How many ways are there to build a tower of height n?

[*Hint: Use composition of generating functions.*]

8-2. Recall that a **Dyck path of length** n is a lattice path from $(0,0)$ to (n,n) that stays above the line $y = x$.)

(a) Find and list the 14 Dyck paths of length 4 and the 14 multiplication schemes for 5 variables.

(b) Use the Catalan bijections from class to determine which Dyck path corresponds to which multiplication scheme.

8-3. Use a bijection to show that sequences $1 \le a_1 \le a_2 \le \cdots \le a_n$ of length n, where each $a_i \le i$ are also counted by the Catalan number C_n. For example, when $n = 3$, the five sequences are 111, 112, 113, 122, and 123.

[*Hint: Look at the boxes to the left of a Dyck path.*]

8-4. Two combinatorial interpretations of the q-binomial coefficients are given on page 130 of the course notes.

(a) Show that for the permutations π of the multiset $\{1^2, 2^3\}$, $\sum_{\pi \in S_{2,3}} q^{\mathsf{inv}(\pi)} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q$.

(b) Show that for the set of lattice paths P from $(0,0)$ to $(2,3)$, $\sum_{P \in \mathcal{P}} q^{\mathsf{area}(P)} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q$.

8-5. Let $\mathcal{C}_n$ denote the set of compositions of n.
For any composition c, define the statistic $\mathsf{parts}(c)$ to be the number of parts of c.
[In other words, if c is the composition $c_1 + c_2 + \cdots + c_k$, then $\mathsf{parts}(c) = k$.]

(a) Compute $f_n(q) = \displaystyle\sum_{c \in \mathcal{C}_n} q^{\mathsf{parts}(c)}$.

(b) Use your answer to part (a) to show directly $\lim_{q \to 1} f_n(q) = 2^{n-1}$.

[*We expect part* (b) *to be true because there are* 2^{n-1} *compositions of* n, *and part* (a) *is constructing a* q*-analog.*]