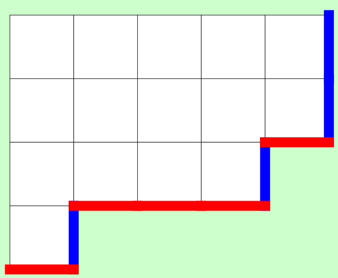
Partitions

The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has λ_i boxes in row i.

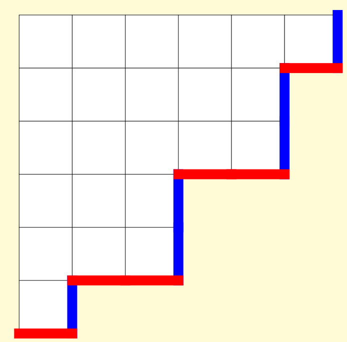
(James, Kerber) Create an **abacus diagram** from the boundary of λ.

Abacus: Function $a : \mathbb{Z} \to \{\bullet, \text{␣}\}$. (Equivalence class...)

Partitions correspond to abacus diagrams.

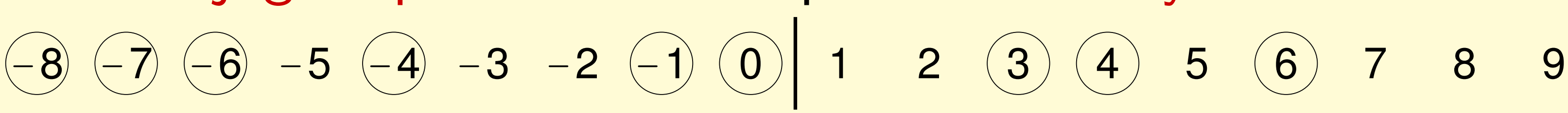


Partition



Self-conjugate partition

Self-conjugate partitions correspond to anti-symmetric abaci.

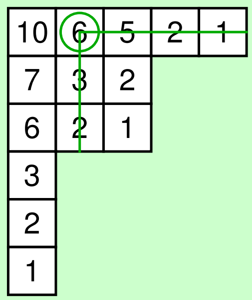


Core partitions

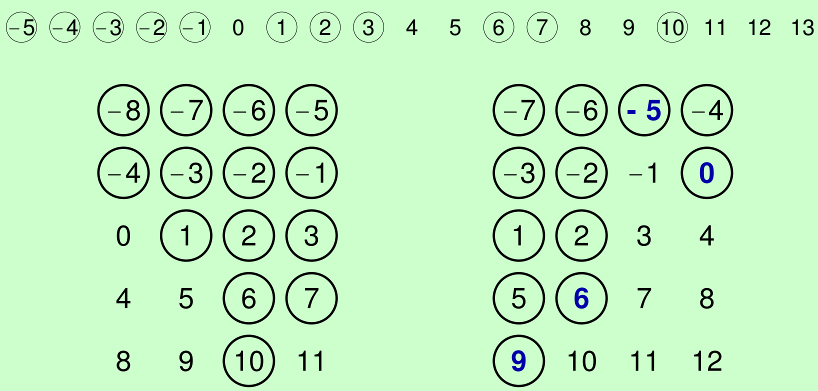
The **hook length** of a box = # boxes below + # boxes to right + box

λ is a ***t*-core** if no boxes have hook length t $\longleftrightarrow$ ***t*-flush** abacus

t-core partition

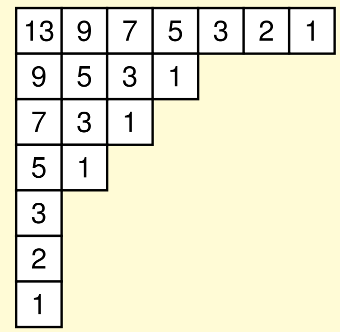


t-flush abacus (in runners)

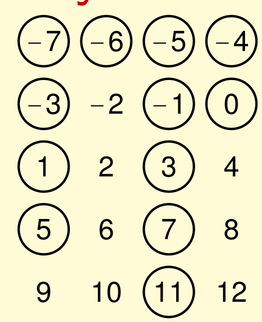


Normalized Balanced

Self-conj. *t*-core partition



t-flush antisymmetric abacus



Antisymmetry about $t/t+1$.

(Discuss defining beads, reading off hooks....)

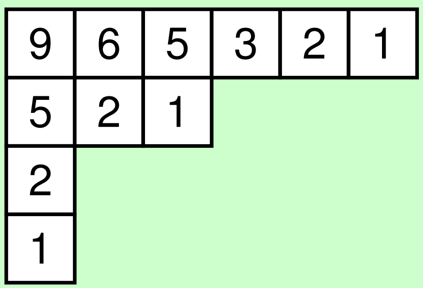
Simultaneity

Of interest: Partitions that are **both** s-core **and** t-core. $(s,t)=1$

- Abaci that are both s-flush and t-flush.

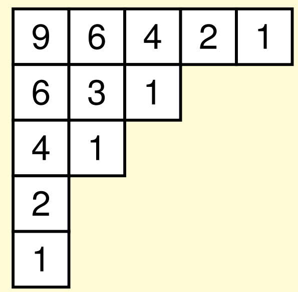
There are infinitely many (self-conjugate) t-core partitions.

(s,t)-core partitions



(Anderson, 2002):
$\#$ (s,t)-core partitions
$\frac{1}{s+t}\binom{s+t}{s}$

Self-conj. (s,t)-core partitions



(Ford, Mai, Sze, 2009):
$\#$ self-conj. (s,t)-core partitions
$\binom{s'+t'}{s'}$
where $s'=\left\lfloor \frac{s}{2}\right\rfloor$ and $t'=\left\lfloor \frac{t}{2}\right\rfloor$

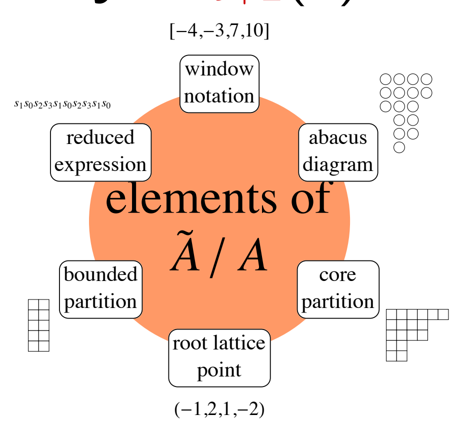
Core partitions in the literature

- **Representation Theory:** (origin)
  - Nakayama conjecture, proved by Brauer & Robinson 1947 says t-cores label t-blocks of irreducible modular representations for S_n.
- **Number Theory:**
  - Let $c_t(n) = \#$ of t-core partitions of n.
  - In 1976, Olsson proved $\displaystyle\sum_{n\geq 0} c_t(n)x^n = \prod_{n\geq 1} \frac{(1-x^{nt})^t}{1-x^n}$

  Numerical properties of $c_t(n)$?

  - 1996: Granville & Ono proved **positivity:** $c_t(n) > 0$ $(t \geq 4)$.
  - 1999: Stanton conjectured **monotonicity:** $c_{t+1}(n) \geq c_t(n)$
  - 2012: R. Nath & I conjectured **monotonicity:** $sc_{t+2}(n) \geq sc_t(n)$
- **Modular forms:** g.f. related to Dedekind's η-fcn, a m.f. of wt. $1/2$.
- **Group Theory:** By Lascoux 2001, t-cores $\longleftrightarrow$ coset reps in $\widetilde{S}_t/S_t$

  Group actions on combinatorial objects!!!!



Reflection Groups

The combinatorics of **groups**:

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - ALTHOUGH, it is **not** the case that $w_1 w_2 = w_2 w_1$.
- There is an identity element (id) & Every element has an inverse.
- Think: (Non-zero real numbers) or (invertible $n \times n$ matrices.)

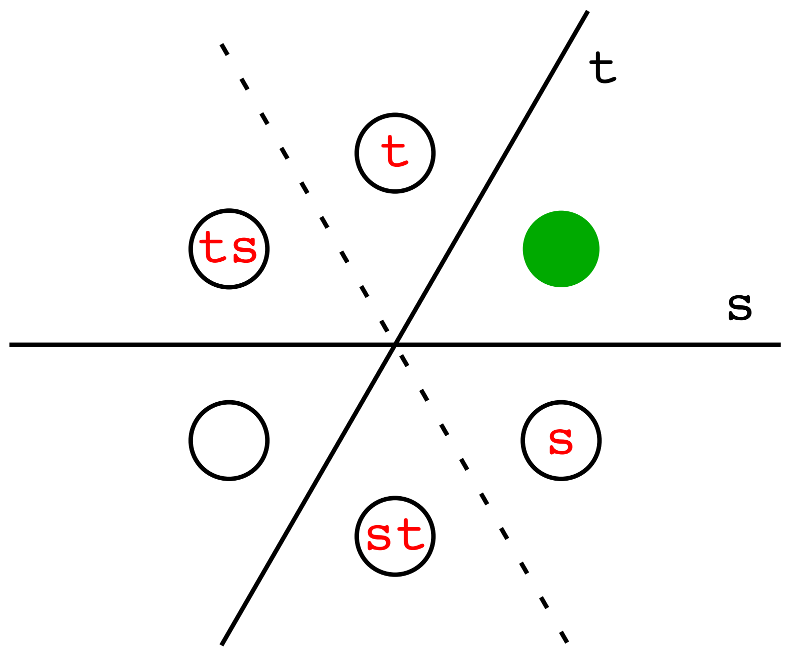
We will talk about **reflection groups**. (With nice pictures)

- W is **generated** by a set of generators $S = \{s_1, s_2, \ldots, s_k\}$.
  - Every $w \in W$ can be written as a product of generators.
- Along with a set of **relations**.
  - These are rules to convert between expressions.
  - $s_i^2 = \text{id}$. —and— $(s_i s_j)^{\text{power}} = \text{id}$.

For example, $w = s_3 s_2 s_1 s_1 s_2 s_4 = s_3 s_2 \text{id} s_2 s_4 = s_3 \text{id} s_4 = s_3 s_4$

Reflection Groups

- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. $(s_i^2 = \text{id})$

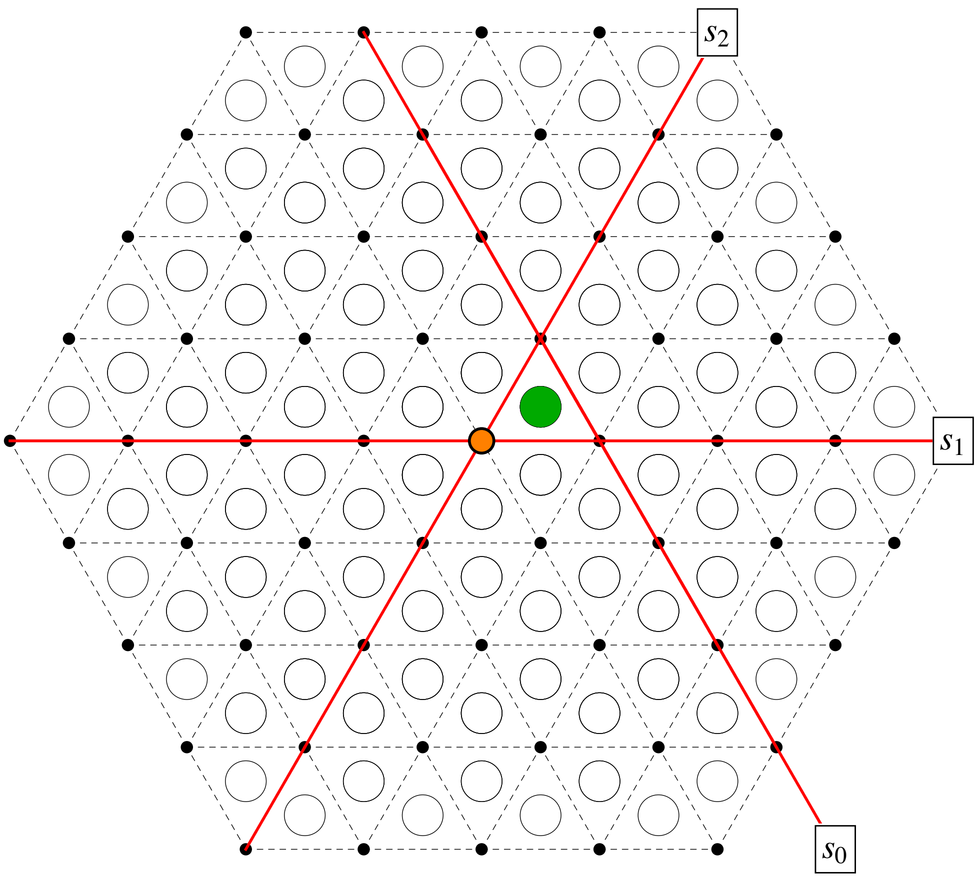


- When the angle between H_s and H_t is $\frac{\pi}{3}$, relation is $(st)^3 = \text{id}$.
- The group depends on the placement of the hyperplanes. $|S| = 6$.

Infinite Reflection Groups

An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.

- Add a new generator s_0 and a new affine hyperplane H_0.



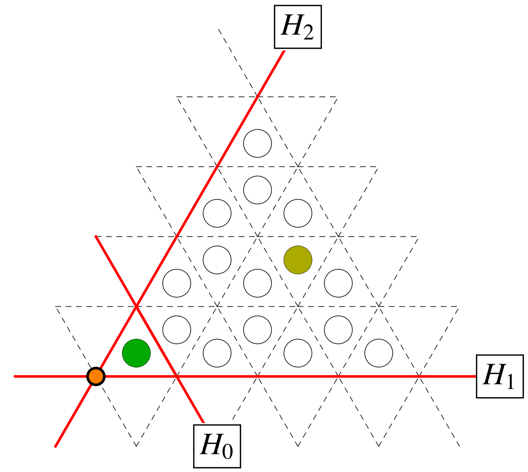
Elements generated by $\{s_0, s_1, s_2\}$ correspond to **alcoves** here.

Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **Permutation.** Similar to writing finite permutations as 312.
- **Abacus diagram.** Columns of numbers.
- **Core partition.** Hook length condition.
- **Bounded partition.** Part size bounded.
- **Others!** Lattice path, order ideal, etc.

They all play nicely with each other.



Coordinates:

3	1
1	

Word: $s_0 s_1 s_2 s_1 s_0$

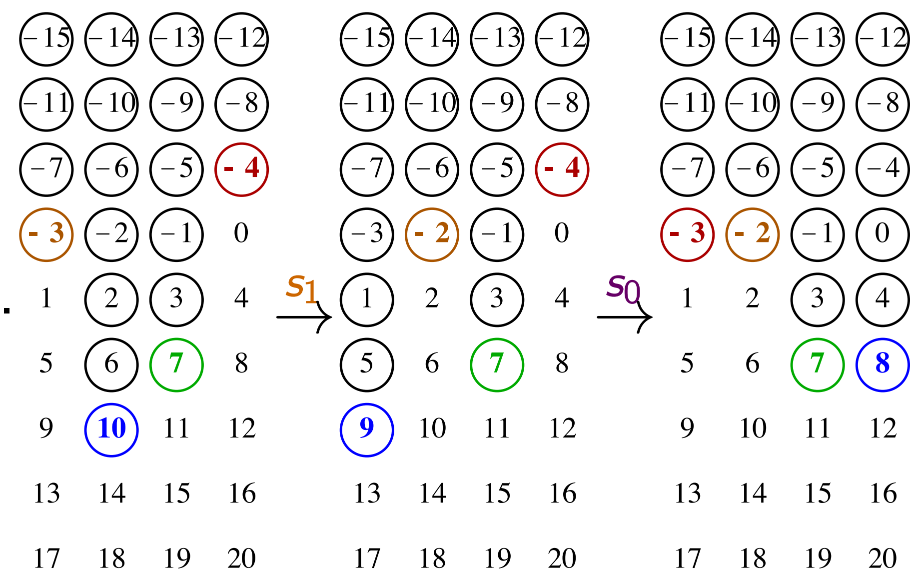
Permutation:

$$(-3, 2, 7)$$

An abacus model for affine permutations

(James and Kerber, 1981) Given an affine permutation $[w_1, \ldots, w_n]$,

- Place integers in n *runners*.
- Circled: *beads*. Empty: *gaps*
- Create an abacus where each runner has a lowest bead at w_i.



Example: $[-4, -3, 7, 10]$

- **Generators act nicely.**
- s_i interchanges runners $i \leftrightarrow i+1$. ($s_1 : 1 \leftrightarrow 2$)
- s_0 interchanges runners 1 and n (with shifts) ($s_0 : 1 \overset{\text{shift}}{\leftrightarrow} 4$)

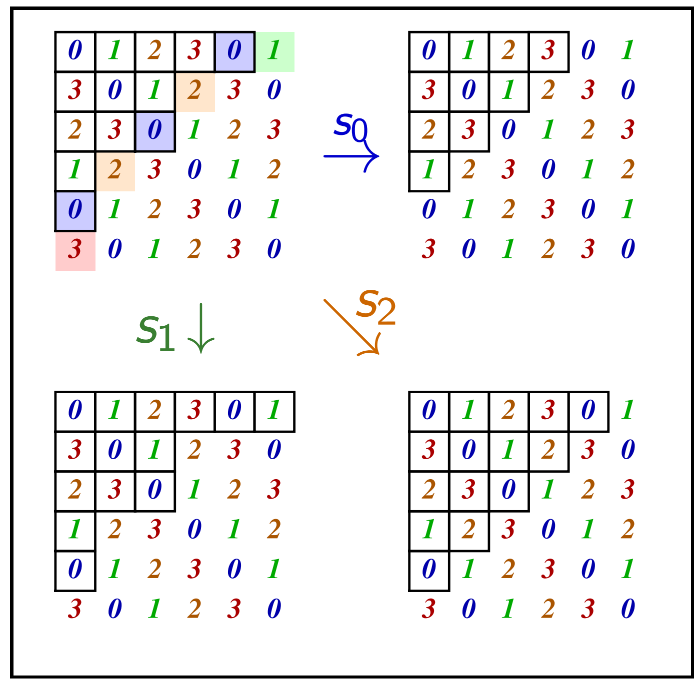
Action of generators on the core partition

- Label the boxes of λ with residues.
- s_i acts by adding or removing boxes with residue i.

0	1	2	3	0	1
3	0	1	2	3	0
2	3	0	1	2	3
1	2	3	0	1	2
0	1	2	3	0	1
3	0	1	2	3	0

Example. $\lambda = (5, 3, 3, 1, 1)$ is a 4-core.

- has removable 0 boxes
- has addable 1, 2, 3 boxes.

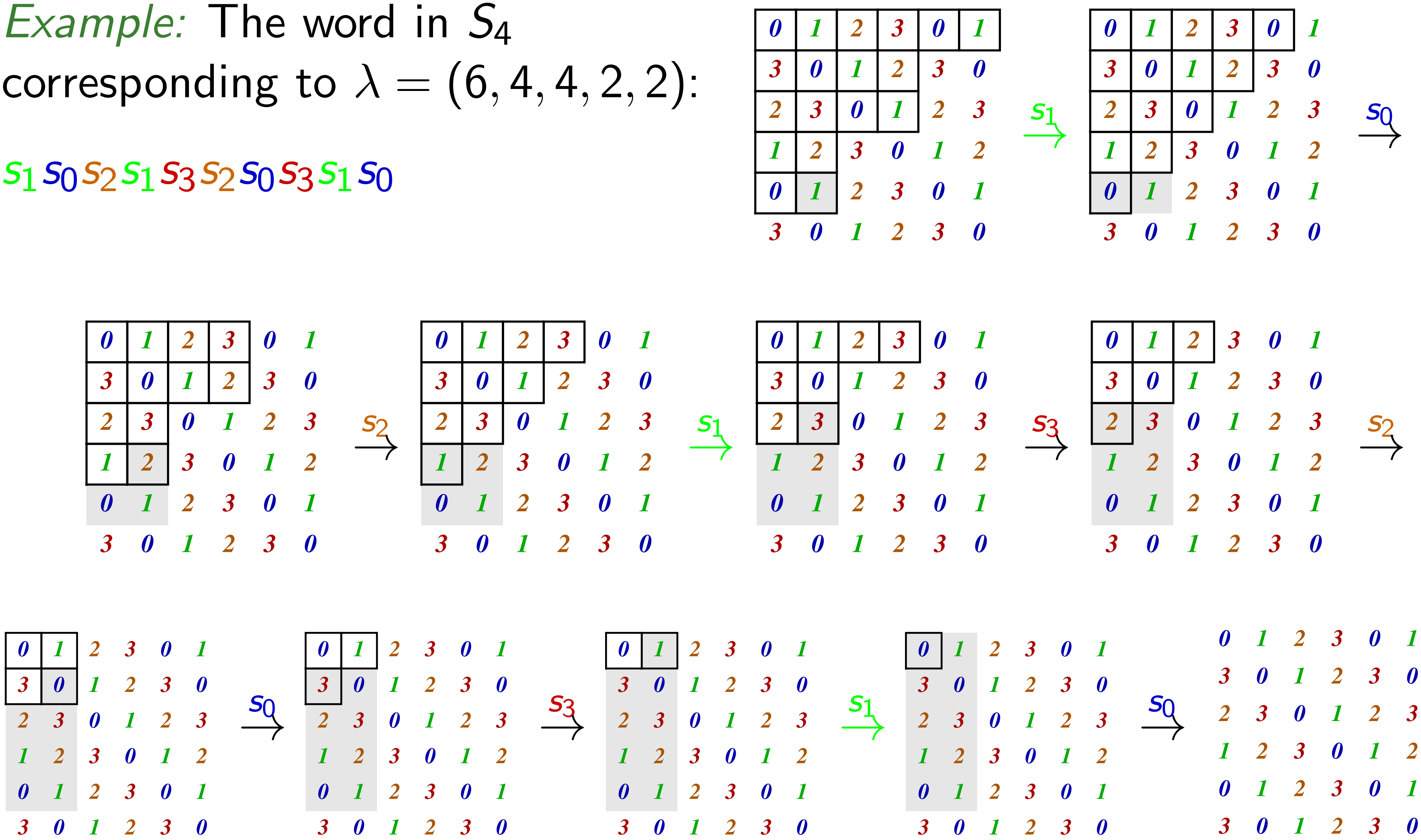


Idea: We can use this to figure out a *word* for w.

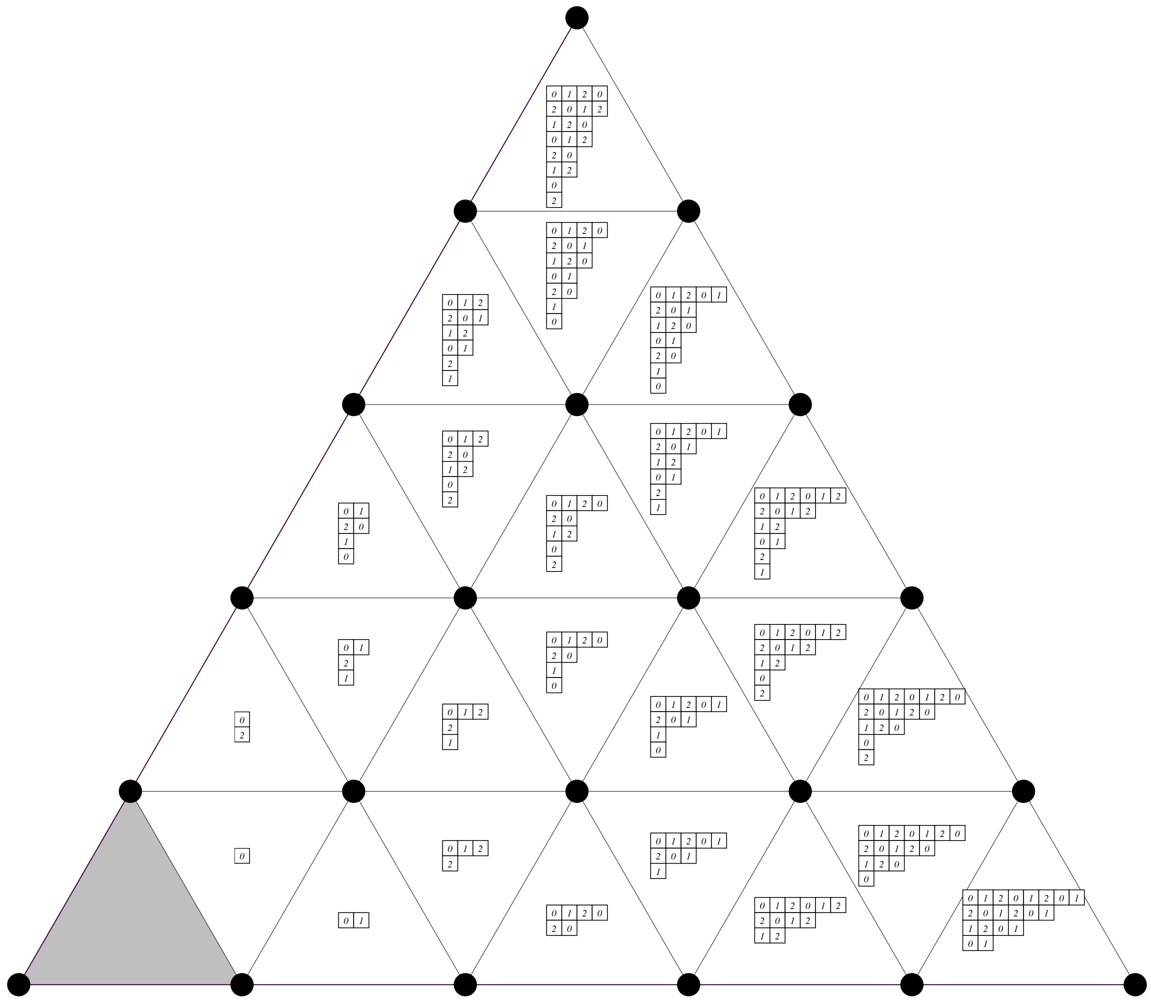
Finding a word for an affine permutation.

Example: The word in S_4 corresponding to $\lambda = (6, 4, 4, 2, 2)$:

$s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$

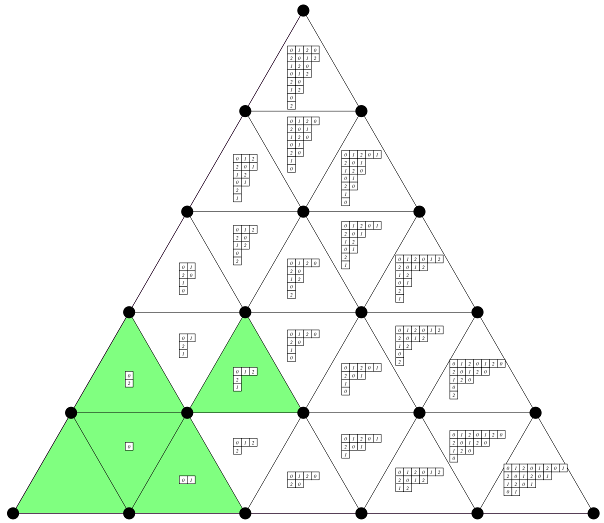


The bijection between cores and alcoves



Simultaneous core partitions

How many partitions are both 2-cores and 3-cores? **2**.



How many partitions are both 3-cores and 4-cores? **5**.

How many simultaneous 4/5-cores? **14**.

How many simultaneous 5/6-cores? **42**.

How many simultaneous $n/(n+1)$-cores? C_n!

Jaclyn Anderson proved that the number of s/t-cores is $\frac{1}{s+t}\binom{s+t}{s}$.

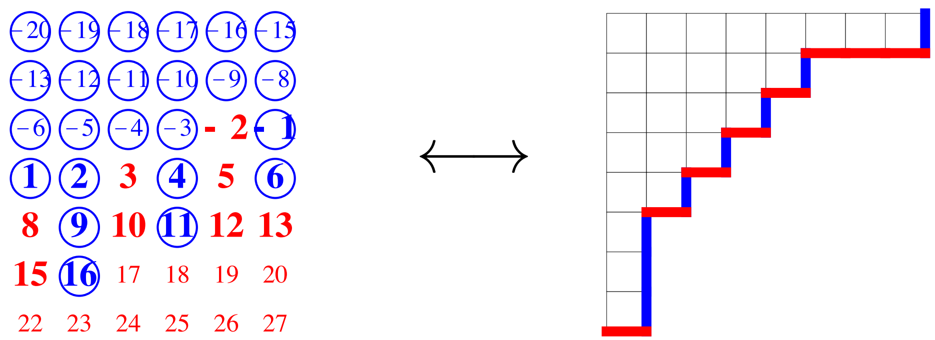
The number of 3/7-cores is $\frac{1}{10}\binom{10}{3} = \frac{1}{10}\frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 12$.

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.

Research Questions

★ Can we extend combinatorial interps to other reflection groups?

- Yes! Involves self-conjugate partitions.
- Article (28 pp) published in *Journal of Algebra.* (2012)
- Joint with Brant Jones, James Madison University.



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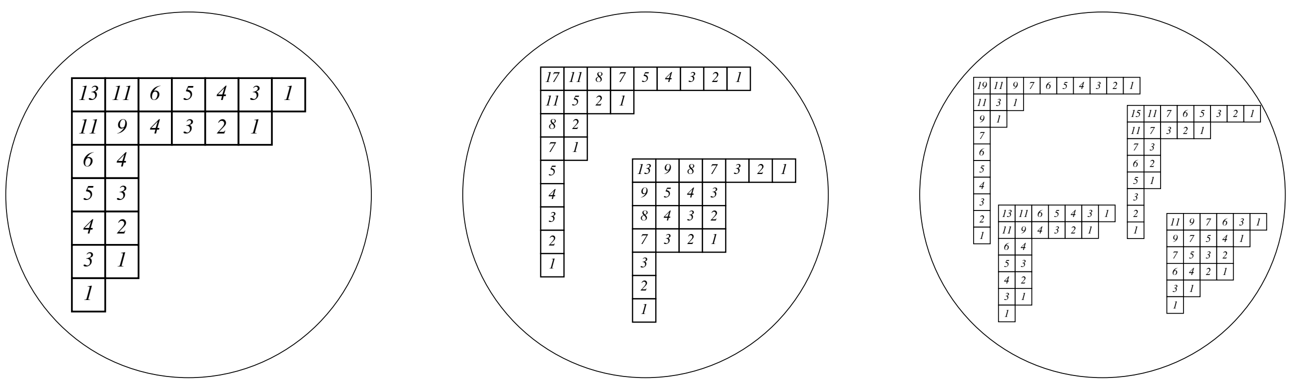
★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.

4–cores of 22 6–cores of 22 8–cores of 22



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★ Properties of simultaneous core partitions. (Formula: $\frac{1}{s+t}\binom{s+t}{s}$)
- **Question.** What is the average size of an (s, t)-core partition?
- *Progress:* Answer: $(s+t+1)(s-1)(t-1)/24$. Proof?
- **Question:** Is there a core statistic for a q-analog of $\frac{1}{s+t}\binom{s+t}{s}$?
- *Progress:* m-Catalan number C_3 through $(3, 3m+1)$-cores.
- (s, t)-cores $\longleftrightarrow$ certain lattice paths. Statistics galore!

★ Happy to have students who would like to do research!