Catalan Numbers

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
1	1	2	5	14	42	132	429	1430	4862	16796

On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Catalan Numbers

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
1	1	2	5	14	42	132	429	1430	4862	16796

On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Richard Stanley has compiled a list of combinatorial interpretations of Catalan numbers. As of 5/13, numbered (a) to (z), ... (a^8) to (y^8).207

Catalan Numbers

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
1	1	2	5	14	42	132	429	1430	4862	16796

On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Richard Stanley has compiled a list of combinatorial interpretations of Catalan numbers. As of 5/13, numbered (a) to (z), … (a^8) to (y^8).207

triangulations of an $(n+2)$-gon	lattice paths from $(0, 0)$ to (n, n) above $y = x$
sequences with n $+1$'s, n -1's with positive partial sums	multiplication schemes to multiply $n + 1$ numbers

Catalan Number Interpretations

When $n = 3$, there are $c_3 = 5$ members of these families of objects:

1. Triangulations of an $(n + 2)$-gon

Catalan Number Interpretations

When $n = 3$, there are $c_3 = 5$ members of these families of objects:

1. Triangulations of an $(n + 2)$-gon

2. Lattice paths from $(0, 0)$ to (n, n) staying above $y = x$

Catalan Number Interpretations

When $n = 3$, there are $c_3 = 5$ members of these families of objects:

1. Triangulations of an $(n + 2)$-gon
2. Lattice paths from $(0, 0)$ to (n, n) staying above $y = x$
3. Sequences of length $2n$ with n $+1$'s and n -1's such that every partial sum is ≥ 0

Catalan Number Interpretations

When $n = 3$, there are $c_3 = 5$ members of these families of objects:

1. Triangulations of an $(n + 2)$-gon
2. Lattice paths from $(0, 0)$ to (n, n) staying above $y = x$
3. Sequences of length $2n$ with n $+1$'s and n -1's such that every partial sum is ≥ 0
4. Ways to multiply $n + 1$ numbers together two at a time.

Catalan Bijections

We claim that these objects are all counted by the Catalan numbers. So there should be bijections between the sets!

Catalan Bijections

We claim that these objects are all counted by the Catalan numbers. So there should be bijections between the sets!

Bijection 1: $\boxed{\text{triangulations of an } (n+2)\text{-gon}} \longleftrightarrow \boxed{\text{multiplication schemes to multiply } n+1 \text{ numbers}}$

Rule: Label all but one side of the $(n+2)$-gon in order. Work your way in from the outside to label the interior edges of the triangulation: When you know two sides of a triangle, the third edge is the product of the two others. Determine the mult. scheme on the last edge.

Catalan Bijections

Bijection 2: $\boxed{\begin{array}{c}\text{multiplication schemes}\\ \text{to multiply } n+1 \ \#\text{s}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{seqs with } n \ {+1}\text{'s, } n \ {-1}\text{'s}\\ \text{with positive partial sums}\end{array}}$

Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by $+1$'s and the parentheses by -1's.

Catalan Bijections

Bijection 3: $\boxed{\begin{array}{c}\text{seqs with } n \ +1\text{'s, } n \ -1\text{'s} \\ \text{with positive partial sums}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{lattice paths } (0,0) \text{ to} \\ (n,n) \text{ above } y = x\end{array}}$

A sequence of $+$'s and $-$'s converts to a sequence of N's and E's, which is a path in the lattice.

Catalan Numbers

The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the generating function equation that $C(x)$ satisfies:

$$C(x) = 1 + xC(x)^2.$$

Catalan Numbers

The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the generating function equation that $C(x)$ satisfies:

$$C(x) = 1 + xC(x)^2.$$

Example. $\boxed{\text{triangulations of an } (n+2)\text{-gon}}$

Here, x represents one side of the polygon

Either the triangulation has a side or not.

1. No side: Empty triangulation (of *digon*): x^0.
2. Every other triangulation has one side (x contribution) and is a sequence of two other triangulations $C(x)^2$.

Catalan Numbers

Example. $\boxed{\text{lattice paths } (0,0) \text{ to } (n,n) \text{ above } y = x}$

Here, x represents an up-step down-step pair.

Either the lattice path starts with a vertical step or not.

1. No step: Empty lattice path: x^0.
2. Every other lattice path has one vertical step up from diag. and a first horizontal step returning to diag. (x contribution). "Between the V & H steps" and "after the H step" is a sequence of two lattice paths $C(x)^2$.

Therefore, $C(x) = 1 + xC(x)^2$.

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.
Do we take the positive or negative root? Check $x = 0$.

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.
Do we take the positive or negative root? Check $x = 0$.

Now extract coefficients to prove the formula for c_n.

Claim: $\sqrt{1-4x} = 1 + \sum_{k \geq 1} \frac{-2}{k}\binom{2(k-1)}{k-1} x^k$. (Next slide.)

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.
Do we take the positive or negative root? Check $x = 0$.

Now extract coefficients to prove the formula for c_n.

Claim: $\sqrt{1-4x} = 1 + \sum_{k\geq 1} \frac{-2}{k}\binom{2(k-1)}{k-1}x^k$. (Next slide.)

Conclusion. $\frac{1}{2x}(1 - \sqrt{1-4x}) = -\frac{1}{2x}\sum_{k\geq 1} \frac{-2}{k}\binom{2(k-1)}{k-1}x^k$

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.
Do we take the positive or negative root? Check $x = 0$.

Now extract coefficients to prove the formula for c_n.

Claim: $\sqrt{1-4x} = 1 + \sum_{k\geq 1} \frac{-2}{k}\binom{2(k-1)}{k-1}x^k$. (Next slide.)

Conclusion.
$$\begin{aligned}\tfrac{1}{2x}(1-\sqrt{1-4x}) &= -\tfrac{1}{2x}\sum_{k\geq 1}\tfrac{-2}{k}\binom{2(k-1)}{k-1}x^k \\ &= \sum_{k\geq 1}\tfrac{1}{k}\binom{2(k-1)}{k-1}x^{k-1}\end{aligned}$$

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.
Do we take the positive or negative root? Check $x = 0$.

Now extract coefficients to prove the formula for c_n.

Claim: $\sqrt{1-4x} = 1 + \sum_{k\geq 1} \frac{-2}{k}\binom{2(k-1)}{k-1}x^k$. (Next slide.)

Conclusion.
$$\begin{aligned}\frac{1}{2x}(1-\sqrt{1-4x}) &= -\frac{1}{2x}\sum_{k\geq 1}\frac{-2}{k}\binom{2(k-1)}{k-1}x^k \\ &= \sum_{k\geq 1}\frac{1}{k}\binom{2(k-1)}{k-1}x^{k-1} \\ &= \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n\end{aligned}$$

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.
Do we take the positive or negative root? Check $x = 0$.

Now extract coefficients to prove the formula for c_n.

Claim: $\sqrt{1-4x} = 1 + \sum_{k \geq 1} \frac{-2}{k}\binom{2(k-1)}{k-1} x^k$. (Next slide.)

Conclusion.
$$\begin{aligned} \tfrac{1}{2x}(1 - \sqrt{1-4x}) &= -\tfrac{1}{2x} \sum_{k \geq 1} \tfrac{-2}{k}\tbinom{2(k-1)}{k-1} x^k \\ &= \sum_{k \geq 1} \tfrac{1}{k}\tbinom{2(k-1)}{k-1} x^{k-1} \\ &= \sum_{n \geq 0} \tfrac{1}{n+1}\tbinom{2n}{n} x^n \end{aligned}$$

Therefore, $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$\sqrt{1-4x} = \big((-4x)+1\big)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k$ Expand $\binom{1/2}{k}$

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$\sqrt{1-4x} = \big((-4x)+1\big)^{1/2} = \sum_{k=0}^{\infty} \tbinom{1/2}{k}(-4x)^k \quad \text{Expand } \tbinom{1/2}{k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k \qquad \text{Denom. of } \tfrac{1}{2}$$

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$\sqrt{1-4x} = \big((-4x)+1\big)^{1/2} = \textstyle\sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k \quad \text{Expand } \binom{1/2}{k}$$

$$= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k \quad \text{Denom. of } \tfrac{1}{2}$$

$$= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k \quad \text{Factor } -2\text{'s}$$

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$
\begin{aligned}
\sqrt{1-4x} &= \big((-4x)+1\big)^{1/2} = \textstyle\sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k && \text{Expand } \tbinom{1/2}{k} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k && \text{Denom. of } \tfrac{1}{2} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k && \text{Factor } -2\text{'s} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k 4^k x^k && \text{Simplify; rewrite prod.}
\end{aligned}
$$

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$\begin{aligned}
\sqrt{1-4x} &= \big((-4x)+1\big)^{1/2} = \textstyle\sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k && \text{Expand } \tbinom{1/2}{k} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k && \text{Denom. of } \tfrac{1}{2} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k && \text{Factor } -2\text{'s} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k 4^k x^k && \text{Simplify; rewrite prod.} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-3)\cdot(2k-2)}{k!\cdot 2\cdot 4\cdots(2k-2)} 2^k x^k && \text{Write as factorials}
\end{aligned}$$

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$\begin{aligned}
\sqrt{1-4x} &= \left((-4x)+1\right)^{1/2} = \textstyle\sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k && \text{Expand } \tbinom{1/2}{k} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k && \text{Denom. of } \tfrac{1}{2} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k && \text{Factor } -2\text{'s} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k 4^k x^k && \text{Simplify; rewrite prod.} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-3)\cdot(2k-2)}{k!\cdot 2\cdot 4\cdots(2k-2)} 2^k x^k && \text{Write as factorials} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{(2k-2)!}{k!(2^{k-1})1\cdot 2\cdots(k-1)} 2^k x^k
\end{aligned}$$

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$\sqrt{1-4x} = \left((-4x)+1\right)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k \quad \text{Expand } \binom{1/2}{k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k \quad \text{Denom. of } \tfrac{1}{2}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k \quad \text{Factor } -2\text{'s}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k 4^k x^k \quad \text{Simplify; rewrite prod.}$$

$$= 1 + \sum_{k=1}^{\infty} -\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-3)\cdot(2k-2)}{k!\cdot 2\cdot 4\cdots(2k-2)} 2^k x^k \quad \text{Write as factorials}$$

$$= 1 + \sum_{k=1}^{\infty} -\frac{(2k-2)!}{k!(2^{k-1})1\cdot 2\cdots(k-1)} 2^k x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{-2}{k} \frac{(2k-2)!}{(k-1)!(k-1)!} x^k$$

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$
\begin{aligned}
\sqrt{1-4x} &= \big((-4x)+1\big)^{1/2} = \textstyle\sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k && \text{Expand } \tbinom{1/2}{k} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k && \text{Denom. of } \tfrac{1}{2} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k && \text{Factor } -2\text{'s} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k 4^k x^k && \text{Simplify; rewrite prod.} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-3)\cdot(2k-2)}{k!\cdot 2\cdot 4\cdots(2k-2)} 2^k x^k && \text{Write as factorials} \\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{(2k-2)!}{k!(2^{k-1})1\cdot 2\cdots(k-1)} 2^k x^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{-2}{k}\frac{(2k-2)!}{(k-1)!(k-1)!} x^k \\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{-2}{k}\binom{2(k-1)}{k-1} x^k
\end{aligned}
$$