Catalan Numbers

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
1	1	2	5	14	42	132	429	1430	4862	16796

On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Richard Stanley has compiled a list of combinatorial interpretations of Catalan numbers. As of 5/13, numbered (a) to (z), ... (a^8) to (y^8).

triangulations of an $(n{+}2)$-gon	lattice paths from $(0,0)$ to (n,n) above $y = x$

sequences with n $+1$'s, n -1's with positive partial sums	multiplication schemes to multiply $n+1$ numbers

Catalan Number Interpretations

When $n = 3$, there are $c_3 = 5$ members of these families of objects:
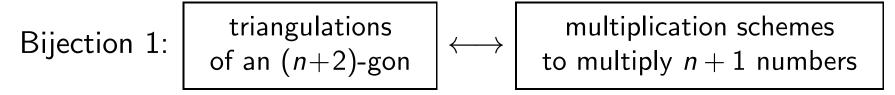
1 Triangulations of an $(n+2)$-gon

2 Lattice paths from $(0,0)$ to (n,n) staying above $y = x$

3 Sequences of length $2n$ with n +1's and n −1's such that every partial sum is ≥ 0

4 Ways to multiply $n+1$ numbers together two at a time.

Catalan Bijections

We claim that these objects are all counted by the Catalan numbers. So there should be bijections between the sets!

Bijection 1:

triangulations of an $(n{+}2)$-gon	$\longleftrightarrow$	multiplication schemes to multiply $n+1$ numbers

Rule: Label all but one side of the $(n+2)$-gon in order. Work your way in from the outside to label the interior edges of the triangulation: When you know two sides of a triangle, the third edge is the product of the two others. Determine the mult. scheme on the last edge.

Catalan Bijections

Bijection 2:
$$\boxed{\begin{array}{c}\text{multiplication schemes}\\\text{to multiply } n+1 \text{ \#s}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{seqs with } n +1\text{'s, } n -1\text{'s}\\\text{with positive partial sums}\end{array}}$$

Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by $+1$'s and the parentheses by -1's.

Catalan Bijections

Bijection 3: $\boxed{\begin{array}{c}\text{seqs with } n +1\text{'s, } n -1\text{'s} \\ \text{with positive partial sums}\end{array}}$ $\longleftrightarrow$ $\boxed{\begin{array}{c}\text{lattice paths } (0,0) \text{ to} \\ (n,n) \text{ above } y = x\end{array}}$

A sequence of $+$'s and $-$'s converts to a sequence of N's and E's, which is a path in the lattice.

Catalan Numbers

The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the generating function equation that $C(x)$ satisfies:

$$C(x) = 1 + xC(x)^2.$$

Example.

triangulations of an $(n{+}2)$-gon

Here, x represents one side of the polygon

Either the triangulation has a side or not.

1. No side: Empty triangulation (of *digon*): x^0.

2. Every other triangulation has one side (x contribution) and is a sequence of two other triangulations $C(x)^2$.

Catalan Numbers

Example. $\boxed{\begin{array}{c} \text{lattice paths } (0,0) \text{ to} \\ (n,n) \text{ above } y = x \end{array}}$ $\quad$ Here, x represents an up-step down-step pair.

Either the lattice path starts with a vertical step or not.

1 No step: Empty lattice path: x^0.

2 Every other lattice path has one vertical step up from diag. and a first horizontal step returning to diag. (x contribution). "Between the V & H steps" and "after the H step" is a sequence of two lattice paths $C(x)^2$.

Therefore, $C(x) = 1 + xC(x)^2$.

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1 - 4x}}{2x}$.

Do we take the positive or negative root? Check $x = 0$.

Now extract coefficients to prove the formula for c_n.

Claim: $\sqrt{1 - 4x} = 1 + \displaystyle\sum_{k \geq 1} \frac{-2}{k} \binom{2(k-1)}{k-1} x^k$. (Next slide.)

Conclusion.
$$
\begin{aligned}
\frac{1}{2x}\left(1 - \sqrt{1 - 4x}\right) &= -\frac{1}{2x} \sum_{k \geq 1} \frac{-2}{k} \binom{2(k-1)}{k-1} x^k \\
&= \sum_{k \geq 1} \frac{1}{k} \binom{2(k-1)}{k-1} x^{k-1} \\
&= \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n
\end{aligned}
$$

Therefore, $c_n = \dfrac{1}{n+1} \binom{2n}{n}$.

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$\sqrt{1-4x} = \left((-4x)+1\right)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k \quad \text{Expand } \binom{1/2}{k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k \quad \text{Denom. of } \tfrac{1}{2}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k \quad \text{Factor } -2\text{'s}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k 4^k x^k \quad \text{Simplify; rewrite prod.}$$

$$= 1 + \sum_{k=1}^{\infty} -\frac{1\cdot2\cdot3\cdot4\cdots(2k-3)\cdot(2k-2)}{k!\cdot2\cdot4\cdots(2k-2)}2^k x^k \quad \text{Write as factorials}$$

$$= 1 + \sum_{k=1}^{\infty} -\frac{(2k-2)!}{k!(2^{k-1})1\cdot2\cdots(k-1)}2^k x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{-2}{k}\frac{(2k-2)!}{(k-1)!(k-1)!}x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{-2}{k}\binom{2(k-1)}{k-1}x^k$$