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An interpretation of this theorem:

If a_k counts all "A" objects of "size" k, and
b_k counts all "B" objects of "size" k, then
$[x^k]\big(A(x)B(x)\big)$ counts all pairs of objects (A, B) with *total* size k.

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So, $[x^k]B(x)S(x)$ counts pairs of the form $\vee$ w/k total candies.
(some number of big candies, some number of small candies)

Vandermonde's Identity (p. 117)

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

Combinatorial proof

Generating function proof

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Similarly, $\big(A(x)\big)^n = \sum_{k\geq 0}\left(\sum_{i_1+i_2+\cdots+i_n=k} a_{i_1}a_{i_2}\cdots a_{i_n}\right)x^k.$

$[x^k](A(x))^n$ counts *sequences* of objects $(A_1, A_2, \ldots, A_n)$, all of type A, with a total size over all objects of k.

Compositions

A **partition**: $p_1 + p_2 + \cdots + p_k = n$ with $p_1 \geq p_2 \geq \cdots \geq p_k$.

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$$c_1 + \cdots + c_\ell \mapsto \begin{cases} c_1 + \cdots + c_{\ell-1} + (c_\ell - 1) & \text{if } c_\ell > 1 \\ c_1 + \cdots + c_{\ell-1} & \text{if } c_\ell = 1 \end{cases}$$

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Therefore, the number of comp's of $k+1$ is $2^{k+1-1} = 2^k$, as desired.

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For a general composition with $g_0 = 0$,

$$F(G(x)) = \sum_{n\geq 0} f_n G(x)^n = f_0 + f_1 G(x) + f_2 G(x)^2 + f_3 G(x)^3 + \cdots .$$

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Interpreting $\frac{1}{1-G(x)} = 1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$:

Recall: The generating function $G(x)^n$ counts sequences of objects $(G_1, G_2, \ldots, G_n)$, each of type G, and the coefficient $[x^k]\big(G(x)^n\big)$ counts those n-sequences that have total size equal to k.

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Conclusion: As long as $g_0 = 0$, then $1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$ counts sequences of **any length** of objects of type G, and the coefficient $[x^k]\frac{1}{1-G(x)}$ counts those that have total size equal to k.

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Think: A composition of generating functions equals
a composition. of. generating. functions.

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Solution. A composition of k corresponds to a sequence $(i_1, \ldots, i_\ell)$ of positive integers (of any length) that sums to k.

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So the number of compositions of n is

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And the generating function for such a military breakdown is

$$H(x) = \frac{1}{1 - G(x)} = \frac{1 - 2x + x^2}{1 - 3x + x^2}$$