Multiplying two generating functions (Convolution)

Let $A(x) = \sum_{k\geq 0} a_k x^k$ and $B(x) = \sum_{k\geq 0} b_k x^k$.

Question: What is the coefficient of x^k in $A(x)B(x)$?

When expanding the product $A(x)B(x)$ we multiply terms $a_i x^i$ in A by terms $b_j x^j$ in B. This product contributes to the coefficient of x^k in AB only when ______________________________.

Therefore, $A(x)B(x) = \sum_{k\geq 0} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k$

An interpretation of this theorem:

If a_k counts all "A" objects of "size" k, and
b_k counts all "B" objects of "size" k, then
$[x^k]\big(A(x)B(x)\big)$ counts all pairs of objects (A, B) with *total* size k.

A Halloween Multiplication

Example. In how many ways can we fill a halloween bag w/30 candies, where for each of 20 BIG candy bars, we can choose at most one, and for each of 40 different small candies, we can choose as many as we like?

Big candy g.f.: $B(x) = (1+x)^{20} = \sum_{k=0}^{\infty} \binom{20}{k} x^k.$

Small candy g.f.: $S(x) = \frac{1}{(1-x)^{40}} = \sum_{k=0}^{\infty} \left(\!\binom{40}{k}\!\right) x^k.$

Total g.f.: $B(x)S(x) = \sum_{k=0}^{\infty} \left[\sum_{i=0}^{k} \binom{20}{i} \left(\!\binom{40}{k-i}\!\right) \right] x^k$

Conclusion: $[x^{30}]B(x)S(x) = \sum_{i=0}^{30} \binom{20}{i} \left(\!\binom{40}{30-i}\!\right)$

b_k counts (k big candies)

s_k counts (k small candies)

So, $[x^k]B(x)S(x)$ counts pairs of the form $\vee$ w/k total candies.
(some number of big candies, some number of small candies)

Vandermonde's Identity (p. 117)

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

Combinatorial proof

Generating function proof

Multiplying two generating functions

Example. What is the coefficient of x^7 in $\frac{x^3(1+x)^4}{(1-2x)}$?

Powers of generating functions

A special case of convolution gives the coefficients of powers of a g.f.:

$$(A(x))^2 = \sum_{k\geq 0}\left(\sum_{i=0}^{k} a_i a_{k-i}\right)x^k = \sum_{k\geq 0}\left(\sum_{i_1+i_2=k} a_{i_1} a_{i_2}\right)x^k.$$

Similarly, $(A(x))^n = \sum_{k\geq 0}\left(\sum_{i_1+i_2+\cdots+i_n=k} a_{i_1} a_{i_2}\cdots a_{i_n}\right)x^k.$

$[x^k](A(x))^n$ counts *sequences* of objects $(A_1, A_2, \ldots, A_n)$, all of type A, with a total size over all objects of k.

Compositions

A **partition**: $p_1 + p_2 + \cdots + p_k = n$ with $p_1 \geq p_2 \geq \cdots \geq p_k$.

A **composition**: $c_1 + c_2 + \cdots + c_k = n$ with no restrictions.

Example. The eight compositions of 4 are:

4 3+1 1+3 2+2 2+1+1 1+2+1 1+1+2 1+1+1+1

Question: How many compositions of n are there?

Answer: 2^{n-1}, which we can prove by induction on n:

Base case. There is one composition of 1.

Step. Suppose that the number of compositions of k is 2^{k-1}.

Create a function from $\mathcal{C}_{k+1} \to \mathcal{C}_k$ that sends

$$c_1 + \cdots + c_\ell \mapsto \begin{cases} c_1 + \cdots + c_{\ell-1} + (c_\ell - 1) & \text{if } c_\ell > 1 \\ c_1 + \cdots + c_{\ell-1} & \text{if } c_\ell = 1 \end{cases}$$

This is a two-to-one function. (Every comp. in $\mathcal{C}_k$ has two preimages.)

Therefore, the number of comp's of $k+1$ is $2^{k+1-1} = 2^k$, as desired.

Compositions of Generating Functions

Question: Let $F(x) = \sum_{n\geq 0} f_n x^n$ and $G(x) = \sum_{n\geq 0} g_n x^n$.
What can we learn about the composition $H(x) = F\big(G(x)\big)$?

Investigate $F(x) = 1/(1-x)$.

$$H(x) = F(G(x)) = \frac{1}{1-G(x)} = 1 + G(x) + G(x)^2 + G(x)^3 + \cdots .$$

- This is an infinite sum of (likely infinite) power series.
- The constant term h_0 of $H(x)$ only makes sense if ______ .
- This implies that x^n divides $G(x)^n$.
  Hence, there are at most $n-1$ summands which contain x^{n-1}.
  We conclude that the infinite sum makes sense.

For a general composition with $g_0 = 0$,

$$F(G(x)) = \sum_{n\geq 0} f_n G(x)^n = f_0 + f_1 G(x) + f_2 G(x)^2 + f_3 G(x)^3 + \cdots .$$

Compositions. of. Generating Functions.

Interpreting $\dfrac{1}{1-G(x)} = 1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$:

Recall: The generating function $G(x)^n$ counts sequences of objects $(G_1, G_2, \ldots, G_n)$, each of type G, and the coefficient $[x^k]\big(G(x)^n\big)$ counts those n-sequences that have total size equal to k.

Conclusion: As long as $g_0 = 0$, then $1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$ counts sequences of **any length** of objects of type G, and the coefficient $[x^k]\frac{1}{1-G(x)}$ counts those that have total size equal to k.

Alternatively: Interpret $[x^k]\frac{1}{1-G(x)}$ thinking of k as this *total size*. First, find **all ways** to break down k into integers $i_1 + \cdots + i_\ell = k$. Then create **all sequences** of objects of type G in which object j has size i_j.

Think: A composition of generating functions equals
a composition. of. generating. functions.

An Example, Compositions

Example. How many compositions of k are there?

Solution. A composition of k corresponds to a sequence $(i_1, \ldots, i_\ell)$ of positive integers (of any length) that sums to k.

The objects in the sequence are positive integers; we need the g.f. that counts how many positive integers with "size i".

What does size correspond to?

How many have value i? Exactly one: the number i.

So the generating function for our objects is
$G(x) = 0 + 1x^1 + 1x^2 + 1x^3 + 1x^4 + \cdots =$ ____________________.

We conclude that the generating function for compositions is
$H(x) = \frac{1}{1-G(x)} =$

So the number of compositions of n is

A Composition Example

Example. How many ways are there to take a line of k soldiers, divide the line into non-empty platoons, and from each platoon choose one soldier in that platoon to be a leader?

Solution. A soldier assignment corresponds to a sequence of platoons of size $(i_1, \ldots, i_\ell)$.

Given i soldiers in a platoon, in how many ways can we assign the platoon a leader? ___________

Therefore $G(x) =$

And the generating function for such a military breakdown is

$$H(x) = \frac{1}{1 - G(x)} = \frac{1 - 2x + x^2}{1 - 3x + x^2}$$