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- The number of pears is zero or one.

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- The number of pears is zero or one.

Strategy. Write down a power series for each piece of fruit, multiply them together, and extract the coefficient of x^k.

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$$= \left[x(1+x)\left(1+x+x^2\right)\right] \cdot \left[x\left(1-x+x^2\right)^2(1+x)\left(1+x+x^2\right)\right]$$

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Die F: $\{1, 2, 2, 3, 3, 4\}$ and die G: $\{1, 3, 4, 5, 6, 8\}$

Solving recurrence relations

Example. Determine a formula for the entries of the sequence $\{a_k\}_{k\geq 0}$ that satisfies $a_0 = 0$ and the recurrence $a_{k+1} = 2a_k + 1$ for $k \geq 0$.

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Solution. Use generating functions: define $A(x) = \sum_{k\geq 0} a_k x^k$.

Step 1: Multiply both sides of the recurrence by x^{k+1} and sum over all k:

$$\sum_{k\geq 0} a_{k+1}x^{k+1} = \sum_{k\geq 0}(2a_k + 1)x^{k+1}$$

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Step 2: Massage the sums to find copies of $A(x)$.
LHS: Re-index, find missing term; RHS: separate into pieces.

$$\sum_{k\geq 1} a_k x^k = \sum_{k\geq 0} 2a_k x^{k+1} + \sum_{k\geq 0} x^{k+1}$$

Conversion to functions of $A(x)$:

Solving recurrence relations

Step 3: Solve for the compact form of $A(x)$.

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Step 4: Extract the coefficients.

When the degree of the numerator is smaller than the degree of the denominator, we can use partial fractions to determine an expression for $A(x)$ of the form:

$$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-x}$$

Solving gives $A(x) = \frac{1}{1-2x} + \frac{-1}{1-x}$; each of which can be expanded:

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$$A(x) = \sum_{k\geq 0} 2^k x^k + \sum_{k\geq 0} (-1)\, x^k = \sum_{k\geq 0} (2^k - 1)\, x^k$$

Therefore, $a_k = 2^k - 1$.

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Therefore, $F(x) - x - 0 = x\big(F(x) - 0\big) + x^2F(x)$, so

$$F(x) = \frac{x}{1 - x - x^2}$$

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Practicality: $(1+\sqrt{5})/2 \approx 1.61803$ and $1 \text{ mi} \approx 1.609344 \text{ km}$

Solving recurrence relations with repeated roots

With repeated roots in the denominator, the result is not quite as nice.

Example. Find the partial fraction decomposition of $\frac{x}{(1-2x)^2(1+5x)}$.

Since $(1-2x)^2$ is a repeated root,

$$\frac{x}{(1-2x)^2(1+5x)} = \frac{A}{(1-2x)} + \frac{B}{(1-2x)^2} + \frac{C}{(1+5x)}.$$

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Clearing the denominator gives:

$$x = A(1-2x)(1+5x) + B(1+5x) + C(1-2x)^2.$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 0 + B(1+\frac{5}{2}) + 0$; so $B = \frac{1}{7}$.
When $x = -\frac{1}{5}$, $-\frac{1}{5} = 0 + 0 + C(1+\frac{2}{5})^2$; so $C = \frac{-5}{49}$.

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Example. Let $\{h_k\}_{k\geq 0}$ be a sequence satisfying

$$h_k + h_{k-1} - 16h_{k-2} + 20h_{k-3} = 0,$$

with initial conditions $h_0 = 1$, $h_1 = 1$, and $h_2 = -1$.

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Find the generating function and formula for h_k.

$$\begin{array}{rcl}
h(x) & = & h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots + h_k x^k + \cdots \,, \\
+xh(x) & = & h_0 x + h_1 x^2 + h_2 x^3 + \cdots + h_{k-1} x^k + \cdots \,, \\
-16x^2 h(x) & = & -16h_0 x^2 - 16h_1 x^3 + \cdots - 16h_{k-2} x^k + \cdots \,, \\
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