Example: Fruit baskets

Example. In how many ways we can create a fruit basket with n pieces of fruit, where we have an infinite supply of apples and bananas, with the added constraints:

- The number of apples is even.
- The number of bananas is a multiple of five.
- The number of oranges is at most four.
- The number of pears is zero or one.

Strategy. Write down a power series for each piece of fruit, multiply them together, and extract the coefficient of x^k.

Example: Rolling dice

Example. When two standard six-sided dice are rolled, what is the distribution of the sums that appear?

Solution. The generating function for one die is $D(x) =$

Therefore, the distribution of sums for rolling two dice is

What does $D(1)$ count?

Example. Is it possible to relabel two six-sided dice differently to give the *exact same distribution* of sums?

Solution. Find two generating functions $F(x)$ and $G(x)$ such that $F(x)G(x) = D^2(x)$ and $F(1) = G(1) = 6$. Rearrange the factors:

$$
\begin{aligned}
D(x)^2 &= x^2(1+x)^2\big(1-x+x^2\big)^2\big(1+x+x^2\big)^2.\\
&= \left[x(1+x)\big(1+x+x^2\big)\right]\cdot\left[x\big(1-x+x^2\big)^2(1+x)\big(1+x+x^2\big)\right]\\
&= \left[x+2x^2+2x^3+x^4\right]\cdot\left[x+x^3+x^4+x^5+x^6+x^8\right]
\end{aligned}
$$

Die F: $\{1,2,2,3,3,4\}$ and die G: $\{1,3,4,5,6,8\}$

Solving recurrence relations

Example. Determine a formula for the entries of the sequence $\{a_k\}_{k\geq 0}$ that satisfies $a_0 = 0$ and the recurrence $a_{k+1} = 2a_k + 1$ for $k \geq 0$.

Solution. Use generating functions: define $A(x) = \sum_{k\geq 0} a_k x^k$.

Step 1: Multiply both sides of the recurrence by x^{k+1} and sum over all k:

$$\sum_{k\geq 0} a_{k+1}x^{k+1} = \sum_{k\geq 0}(2a_k + 1)x^{k+1}$$

Step 2: Massage the sums to find copies of $A(x)$.
LHS: Re-index, find missing term; RHS: separate into pieces.

$$\sum_{k\geq 1} a_k x^k = \sum_{k\geq 0} 2a_k x^{k+1} + \sum_{k\geq 0} x^{k+1}$$

Conversion to functions of $A(x)$:

Solving recurrence relations

Step 3: Solve for the compact form of $A(x)$.

$$A(x) = \frac{x}{(1-2x)(1-x)}$$

Step 4: Extract the coefficients.

When the degree of the numerator is smaller than the degree of the denominator, we can use partial fractions to determine an expression for $A(x)$ of the form:

$$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-x}$$

Solving gives $A(x) = \frac{1}{1-2x} + \frac{-1}{1-x}$; each of which can be expanded:

$$A(x) = \sum_{k\geq 0} 2^k x^k + \sum_{k\geq 0} (-1)\, x^k = \sum_{k\geq 0} (2^k - 1)\, x^k$$

Therefore, $a_k = 2^k - 1$.

A closed form for Fibonacci numbers

Example. Solve the recurrence relation $f_{k+2} = f_{k+1} + f_k$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

Solution. Define $F(x) = \sum_{k\geq 0} f_k x^k$. Then,

$$\sum_{k\geq 0} f_{k+2}x^{k+2} = \sum_{k\geq 0}(f_{k+1} + f_k)x^{k+2}$$

$$\sum_{k\geq 0} f_{k+2}x^{k+2} = \sum_{k\geq 0} f_{k+1}x^{k+2} + \sum_{k\geq 0} f_k x^{k+2}$$

$$\sum_{k\geq 0} f_{k+2}x^{k+2} = x\sum_{k\geq 0} f_{k+1}x^{k+1} + x^2\sum_{k\geq 0} f_k x^k$$

$$\sum_{k\geq 2} f_k x^k = x\sum_{k\geq 1} f_k x^k + x^2\sum_{k\geq 0} f_k x^k$$

Therefore, $F(x) - x - 0 = x\big(F(x) - 0\big) + x^2F(x)$, so

$$F(x) = \frac{x}{1 - x - x^2}$$

A closed form for Fibonacci numbers

So the Fibonacci numbers have generating function $x/(1-x-x^2)$.
The roots of $(1-x-x^2)=(1-r_+x)(1-r_-x)$ are $r_\pm=(1\pm\sqrt{5})/2$.
Using partial fractions,

$$F(x)=\frac{1}{\sqrt{5}}\frac{1}{1-r_+x}-\frac{1}{\sqrt{5}}\frac{1}{1-r_-x}$$

Therefore, $\displaystyle\sum_{k\geq 0} f_k x^k=\sum_{k\geq 0}\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k x^k-\sum_{k\geq 0}\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k x^k$

and we conclude that $\displaystyle f_k=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k.$

As $k\to+\infty$, the second term goes to zero, so $\displaystyle f_k\approx\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k$

Practicality: $(1+\sqrt{5})/2\approx 1.61803$ and $1\text{ mi}\approx 1.609344\text{ km}$

Solving recurrence relations with repeated roots

With repeated roots in the denominator, the result is not quite as nice.

Example. Find the partial fraction decomposition of $\frac{x}{(1-2x)^2(1+5x)}$.
Since $(1-2x)^2$ is a repeated root,

$$\frac{x}{(1-2x)^2(1+5x)} = \frac{A}{(1-2x)} + \frac{B}{(1-2x)^2} + \frac{C}{(1+5x)}.$$

Clearing the denominator gives:

$$x = A(1-2x)(1+5x) + B(1+5x) + C(1-2x)^2.$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 0 + B(1+\frac{5}{2}) + 0$; so $B = \frac{1}{7}$.
When $x = -\frac{1}{5}$, $-\frac{1}{5} = 0 + 0 + C(1+\frac{2}{5})^2$; so $C = \frac{-5}{49}$.

Equating the coefficients of x^0, we see $A + B + C = 0$. We conclude $A = \frac{-2}{49}$.

$$\frac{x}{(1-2x)^2(1+5x)} = \frac{-\frac{2}{49}}{(1-2x)} + \frac{\frac{7}{49}}{(1-2x)^2} + \frac{-\frac{5}{49}}{(1+5x)}.$$

Solving recurrence relations with repeated roots

Example. Let $\{h_k\}_{k\geq 0}$ be a sequence satisfying

$$h_k + h_{k-1} - 16h_{k-2} + 20h_{k-3} = 0,$$

with initial conditions $h_0 = 1$, $h_1 = 1$, and $h_2 = -1$.
Find the generating function and formula for h_k.

$$\begin{array}{rcl}
h(x) & = & h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots + h_kx^k + \cdots\ , \\
+xh(x) & = & h_0x + h_1x^2 + h_2x^3 + \cdots + h_{k-1}x^k + \cdots\ , \\
-16x^2h(x) & = & -16h_0x^2 - 16h_1x^3 + \cdots - 16h_{k-2}x^k + \cdots\ , \\
+20x^3h(x) & = & 20h_0x^3 + \cdots + 20h_{k-3}x^k + \cdots\ , \\
\hline
\end{array}$$

Therefore, $h(x) =$

Since $\frac{1}{(1-y)^m} = \sum_{k\geq 0} \left(\!\binom{m}{k}\!\right) y^k$,

we see $\frac{1}{(1-2x)^2} = \sum_{k\geq 0} \binom{k+1}{k}(2x)^k = \sum_{k\geq 0}(k+1)2^kx^k$.

We conclude that $h_k =$