

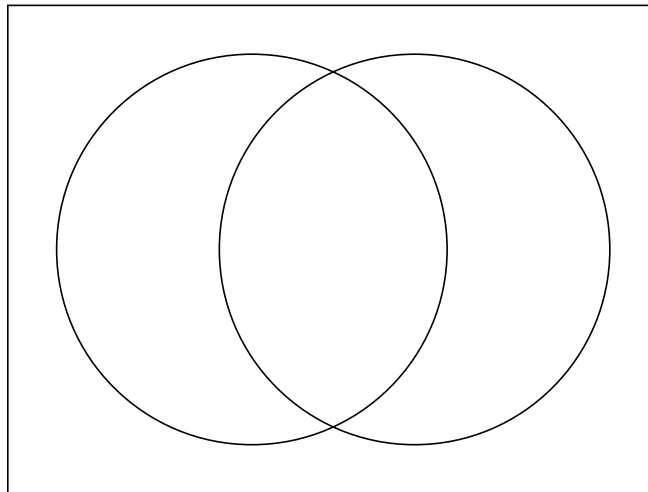
Principle of Inclusion-Exclusion

Example. Suppose that in this class, 14 students play soccer and 11 students play basketball. How many students play a sport?

Solution.

Let S be the set of students who play soccer and B be the set of students who play basketball.

Then, $|S \cup B| = |S| + |B|$ _____.



Principle of Inclusion-Exclusion

When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ (\mathcal{U} for universe) and the sets A_i are *pairwise disjoint*, we have $|A| = |A_1| + \cdots + |A_k|$.

When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ and the A_i are **not** pairwise disjoint, we must apply the **principle of inclusion-exclusion** to determine $|A|$:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| \\ &\quad - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \end{aligned}$$

$$|A_1 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \cdots$$

It may be more convenient to apply inclusion/exclusion where the A_i are *forbidden* subsets of \mathcal{U} , in which case _____.

mmm...PIE

The key to using the principle of inclusion-exclusion is determining the right choice of A_i . The A_i and their intersections should be easy to count and easy to characterize.

Notation: $\pi = p_1 p_2 \cdots p_n$ is the **one-line notation** for a permutation of $[n]$ whose first element is p_1 , second element is p_2 , etc.

Example. How many permutations $p = p_1 p_2 \cdots p_n$ are there in which at least one of p_1 and p_2 are even?

Solution. Let \mathcal{U} be the set of n -permutations.

Let A_1 be the set of permutations where p_1 is even.

Let A_2 be the set of permutations where p_2 is even.

In words, $A_1 \cap A_2$ is the set of n -permutations _____

Now calculate: $|A_1| =$ $|A_2| =$

$|A_1 \cap A_2| =$

Applying PIE: So $|A_1 \cup A_2| =$

mmm...PIE

Example. Find the number of integers between 1 and 1000 that are **not** divisible by 5, 6, or 8.

Solution. Let $\mathcal{U} = \{n \in \mathbb{Z} \text{ such that } 1 \leq n \leq 1000\}$.

Let $A_1 \subset \mathcal{U}$ be the multiples of 5, $A_2 \subset \mathcal{U}$ be the multiples of 6, and $A_3 \subset \mathcal{U}$ be the multiples of 8. We want $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3|$.

In words, $A_1 \cap A_2$ is the set of integers

$A_1 \cap A_3$ is

$A_2 \cap A_3$ is

and $A_1 \cap A_2 \cap A_3$ is the set of integers that are

Now calculate: $|A_1| =$ $|A_2| =$ $|A_3| =$

$|A_1 \cap A_2| =$ $|A_1 \cap A_3| =$ $|A_2 \cap A_3| =$

$|A_1 \cap A_2 \cap A_3| =$

And finally: So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

Combinations with Repetitions

Quick review

- 1 How many ways are there to choose k elements out of the set $\{1 \cdot a_1, 1 \cdot a_2, \dots, 1 \cdot a_n\}$?
- 2 How many ways are there to choose k elements out of the set $\{k \cdot a_1, k \cdot a_2, \dots, k \cdot a_n\}$? (really $\{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_n\}$)

What we would like to calculate is:

In how many ways can we choose k elements out of an arbitrary multiset?

Now, it's as easy as PIE.

Combinations with Repetitions

Example. Determine the number of 10-combinations of the multiset $S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.

Game plan: Let \mathcal{U} be the set of 10-combs of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$. Use PIE to remove the 10-combs that violate the conditions of S

Define A_1 to be 10-combs that include at least ___ a 's.

Define A_2 to be 10-combs that include at least ___ b 's.

Define A_3 to be 10-combs that include at least ___ c 's.

In words, $A_1 \cap A_2$ are those 10-combs that

$A_1 \cap A_3$:

$A_2 \cap A_3$:

$A_1 \cap A_2 \cap A_3$

Now calculate: $|\mathcal{U}| = \quad |A_1| = \quad |A_2| = \binom{3}{5} \quad |A_3| = \binom{3}{4}$

$|A_1 \cap A_2| = 3 \quad |A_1 \cap A_3| = 1 \quad |A_2 \cap A_3| = 0 \quad |A_1 \cap A_2 \cap A_3| = 0$

And finally: So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

Derangements

At a party, 10 partygoers check their hats. They “have a good time”, and are each handed a hat on the way out. In how many ways can the hats be returned so that no one is returned his/her own hat?

This is a **derangement** of ten objects.

Definition: An **n -derangement** is an n -permutation $\pi = p_1 p_2 \cdots p_n$ such that $p_1 \neq 1, p_2 \neq 2, \cdots, p_n \neq n$.

Note: A derangement is a permutation without fixed points $\pi(i) = i$.

Notation: We let D_n be the number of all n -derangements.

When you see D_n , think combinatorially: “The number of ways to return n hats to n people so no one gets his/her own hat back”

Calculating the number of derangements

Example. Calculate D_n .

Solution. Let \mathcal{U} be the set of all n -permutations.

Remove bad permutations using PIE.

For all i from 1 to n , define A_i to be n -perms where $p_i = i$.

In words, $A_i \cap A_j$ are n -perms where

$A_i \cap A_j \cap A_k$ are n -perms where

In general, $A_{i_1} \cap \dots \cap A_{i_k}$ are n -perms with $p_{i_1} = i_1, \dots, p_{i_k} = i_k$.

Now calculate: $|\mathcal{U}| =$ $|A_1| =$ $|A_2| =$

For all i and j , $|A_i \cap A_j| =$

When intersecting k sets, $|A_{i_1} \cap \dots \cap A_{i_k}| =$

Recall: $|A_1 \cup \dots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \dots$

Therefore, $D_n = |\mathcal{U}| - |A_1 \cup \dots \cup A_n| =$

Randomly returning hats

Upon simplification, we see

$$\begin{aligned}
 D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! \\
 &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\
 &= n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right]
 \end{aligned}$$

Recall: Taylor series expansion of e^x :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Plug in $x = -1$ and truncate after n terms to see that

$$e^{-1} \approx \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right]$$

Conclusion: If n people go to a party and the hats are passed back randomly, the probability that no one gets his or her hat back at the party is $D_n/n!$, which is approximately $1/e \approx 37\%$.

Combinatorial proof involving D_n

Recall: The combinatorial interpretation of D_n is: “The number of ways to return n hats to n people so no one gets his/her own hat back”

Example. Prove the following recurrence relation for D_n combinatorially.

$$D_n = (n - 1)(D_{n-2} + D_{n-1})$$

A formula for Stirling numbers (p. 90)

(Careful: change of notation !!)

Recall: $S(n, k) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the number of partitions of the set $[n]$ into exactly k parts, and $k!S(n, k)$ is the number of **onto functions** $[n] \rightarrow [k]$.

Question: What is a formula for $S(n, k)$?

Solution. We will find the number of surjections from $[n]$ to $[k]$.

Use PIE with \mathcal{U} = set of **all** functions from $[n]$ to $[k]$.

We will remove the “bad” functions where the range is not $[k]$.

Define A_i be the set of functions $f : [n] \rightarrow [k]$ where i is not “hit”.

In words, $A_{i_1} \cap \cdots \cap A_{i_j}$ are functions where none of i_1 through i_j are elements of the image.

We calculate: $|\mathcal{U}| = k^n$, $|A_i| = (k - 1)^n$, $|A_i \cap A_j| = (k - 2)^n$

When intersecting j sets, $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k - j)^n$.

Therefore, $k!S(n, k) = \sum_{j=0}^k (-1)^j \binom{k}{j} (k - j)^n$; we conclude

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k - j)^n = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n.$$

A formula for Bell numbers (p. 166)

(Careful: change of notation !!)

Recall: B_n is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

$$\begin{aligned}
 B_n &= \sum_{k \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^k \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n \\
 &= \sum_{k \geq 0} \sum_{j=0}^k \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^k \frac{(-1)^{k-j} j^n}{(k-j)! j!} \\
 &= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)! j!} = \sum_{j \geq 0} \frac{j^n}{j!} \sum_{m \geq 0} \frac{(-1)^m}{(m)!} = \sum_{j \geq 0} \frac{j^n}{j!} \frac{1}{e}
 \end{aligned}$$

Theorem 4.3.3. For any $n > 0$, $B_n = \frac{1}{e} \sum_{j \geq 0} \frac{j^n}{j!}$.

For example, $B_5 = \frac{1}{e} \left(\frac{0^5}{0!} + \frac{1^5}{1!} + \frac{2^5}{2!} + \frac{3^5}{3!} + \frac{4^5}{4!} + \frac{5^5}{5!} + \dots \right) = 52$.