Counting integral solutions

Question: How many non-negative integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$?

- Give some examples of solutions.
- Characterize what solutions look like.
- A combinatorial object with a similar flavor is:

In general, the number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = k$ is ________.

Question: How many **positive** integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$, where $x_4 \geq 3$?

The sum principle

Often it makes sense to break down your counting problem into smaller, disjoint, and easier-to-count sub-problems.

Example. How many integers from 1 to 999999 are palindromes?

Answer: Condition on how many digits.

- Length 1:
- Length 2:
- Length 3:
- Length 4:
- Length 5,6:
- **Total**:

★ Every palindrome between 1 and 999999 is counted once.

This illustrates the **sum principle**:

Suppose the objects to be counted can be broken into k disjoint and exhaustive cases. If there are n_j objects in case j, then there are $n_1 + n_2 + \cdots + n_k$ objects in all.

Counting pitfalls

When counting, there are two common pitfalls:

- Undercounting
  - Often, forgetting cases when applying the sum principle.
  - **Ask:** Did I miss something?
- Overcounting
  - Often, misapplying the product principle.
  - **Ask:** Do cases need to be counted in different ways?
  - **Ask:** Does the same object appear in multiple ways?

Common example: A deck of cards.

There are four suits: Diamond $\diamondsuit$, Heart $\heartsuit$, Club $\clubsuit$, Spade $\spadesuit$.
Each has 13 cards: Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2.

Example. Suppose you are dealt two diamonds between 2 and 10.
In how many ways can the product be even?

Overcounting

Example. In Blackjack you are dealt 2 cards: 1 face-up, 1 face-down. In how many ways can the face-down card be an Ace and the face-up card be a Heart ♡?

Answer: There are __ aces, so there are ___ choices for the down card. There are __ hearts, so there are ____ choices for the up card. By the product principle, there are 52 ways in all.

Except:

Remember to ask: Do cases need to be counted in different ways?

Overcounting

Example. How many 4-lists taken from [9] have at least one pair of adjacent elements equal?

Examples: 1114, 1229, 5555 Non-examples: 1231, 9898.

Strategy:

1. Choose where the adjacent equal elements are. (___ ways)
2. Choose which number they are. (___ ways)
3. Choose the numbers for the remaining elements. (___ ways)

By the product principle, there are ________ ways in all.

Except:

Remember to ask: Does the same object appear in multiple ways?

Counting the complement

Q1: How many 4-lists taken from [9] have **at least one** pair of adjacent elements equal?

—Compare this to—

Q2: How many 4-lists taken from [9] have **no** pairs of adjacent elements equal?

What can we say about:

Q1: **Q2:** **Together:**

Q3:

Strategy: It is sometimes easier to **count the complement**.

Answer to Q3:

Answer to Q2:

Answer to Q1:

Poker hands

Example. When playing five-card poker, what is the probability that you are dealt a full house?

[*Three cards of one type and two cards of another type.*] 5 5 5 K K

Game plan:

- Count the total number of hands.
- Count the number of possible full houses. **# of ways**
  - Choose the denomination of the three-of-a-kind.
  - Choose which three suits they are in.
  - Choose the denomination of the pair.
  - Choose which two suits they are in.
  - Apply the multiplication principle. **Total:**
- Divide to find the probability.

Introduction to Bijections

Key tool: A useful method of proving that two sets A and B are of the same size is by way of a *bijection*.

A **bijection** is a function or rule that pairs up elements of A and B.

Example. The set A of subsets of $\{s_1, s_2, s_3\}$ are in bijection with the set B of binary words of length 3.

Set A:	$\{\emptyset,$	$\{s_1\},$	$\{s_2\},$	$\{s_1, s_2\},$	$\{s_3\},$	$\{s_1, s_3\},$	$\{s_2, s_3\},$	$\{s_1, s_2, s_3\}\}$
Bijection:	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
Set B:	$\{000,$	$100,$	$010,$	$110,$	$001,$	$101,$	$011,$	$111\}$

Rule: Given $a \in A$, (a is a subset), define $b \in B$ (b is a word):
If $s_i \in a$, then letter i in b is 1. If $s_i \notin a$, then letter i in b is 0.

Difficulties:

- Finding the function or rule (requires rearranging, ordering)
- Proving the function or rule (show it **IS** a bijection).

What is a Function?

Reminder: A **function** f from A to B (write $f : A \to B$) is a rule where for each element $a \in A$, $f(a)$ is defined as an element $b \in B$ (write $f : a \mapsto b$).

- A is called the **domain**. (We write $A = \mathrm{dom}(f)$)
- B is called the **codomain**. (We write $B = \mathrm{cod}(f)$)
- The **range** of f is the set of values that f takes on:

$$\mathrm{rng}(f) = \{b \in B : f(a) = b \text{ for at least one } a \in A\}$$

Example. Let A be the set of 3-subsets of $[n]$ and let B be the set of 3-lists of $[n]$. Then define $f : A \to B$ to be the function that takes a 3-subset $\{i_1, i_2, i_3\} \in A$ (with $i_1 \le i_2 \le i_3$) to the word $i_1 i_2 i_3 \in B$.

Question: Is $\mathrm{rng}(f) = B$?

What is a Bijection?

Definition: A function $f : A \to B$ is **one-to-one** (an **injection**) when

For each $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Equivalently,

For each $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

"When the inputs are different, the outputs are different." (picture)

Definition: A function $f : A \to B$ is **onto** (a **surjection**) when

For each $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

"Every output gets hit."

Definition: A function $f : A \to B$ is a **bijection** if it is both one-to-one and onto.

The function from the previous page is ________________________.

What is an example of a function that is onto and not one-to-one?

Proving a Bijection

Example. Use a bijection to prove that $\binom{n}{k} = \binom{n}{n-k}$ for $0 \leq k \leq n$.

Proof. Let A be the set of k-subsets of $[n]$
and let B be the set of $(n-k)$-subsets of $[n]$.

A bijection between A and B will prove $\binom{n}{k} = |A| = |B| = \binom{n}{n-k}$.

Step 1: Find a candidate bijection.

Strategy. Try out a small (enough) example. Try $n = 5$ and $k = 2$.

$$\left\{ \begin{array}{c} \{1,2\},\ \{1,3\} \\ \{1,4\},\ \{1,5\} \\ \{2,3\},\ \{2,4\} \\ \{2,5\},\ \{3,4\} \\ \{3,5\},\ \{4,5\} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \{1,2,3\},\ \{1,2,4\} \\ \{1,2,5\},\ \{1,3,4\} \\ \{1,3,5\},\ \{1,4,5\} \\ \{2,3,4\},\ \{2,3,5\} \\ \{2,4,5\},\ \{3,4,5\} \end{array} \right\}$$

Guess: Let S be a k-subset of $[n]$. Perhaps $f(S) =$ ______.

Proving a Bijection

Step 2: Prove f is well defined.

The function f is well defined. If S is any k-subset of $[n]$, then S^c is a subset of $[n]$ with $n - k$ members. Therefore $f : A \to B$.

Step 3: Prove f is a bijection.

Strategy. Prove that f is both one-to-one and onto.

f is 1-to-1: Suppose that S_1 and S_2 are two k-subsets of $[n]$ such that $f(S_1) = f(S_2)$. That is, $S_1^c = S_2^c$. This means that for all $i \in [n]$, then $i \notin S_1$ if and only if $i \notin S_2$. Therefore $S_1 = S_2$ and f is 1-to-1.

f is onto: Suppose that $T \in B$ is an $(n-k)$-subset of $[n]$. We must find a set $S \in A$ satisfying $f(S) = T$. Choose $S =$ ______. Then $S \in A$ (why?), and $f(S) = S^c = T$, so f is onto.

We conclude that f is a bijection and therefore, $\binom{n}{k} = \binom{n}{n-k}$.

Using the Inverse Function

When $f : A \to B$ is 1-to-1, we can define f's **inverse**.

We write f^{-1}, and it is a function from $\mathrm{rng}(f)$ to A.

It is defined via f. If $f : a \mapsto b$, then $f^{-1} : b \mapsto a$.

Caution: When f is a function from A to B, f^{-1} might not be a function from B to A.

Theorem. Suppose that A and B are finite sets and that $f : A \to B$ is a function. If f^{-1} is a function with domain B, then f is a bijection.

Proof. Since f^{-1} is only defined when f is 1-to-1, we need only prove that f is onto. Suppose $b \in B$. By assumption, $f^{-1}(b) \in A$ exists and $f(f^{-1}(b)) = b$. So f is onto, and is a bijection.

Consequence: An alternative method for proving a bijection is:

- Find a rule $g : B \to A$ which always takes $f(a)$ back to a.
- Verify that the domain of g is *all of B*.

Using the Inverse Function

Example. There exists as many even-sized subsets of $[n]$ as odd-sized subsets of $[n]$.

$$\begin{array}{ll} \text{even:} & \{\ \emptyset,\ \{s_1, s_2\}, \{s_1, s_3\},\ \{s_2, s_3\}\ \} \\ \text{odd:} & \{\{s_1\},\ \{s_2\},\ \{s_3\},\ \{s_1, s_2, s_3\}\} \end{array}$$

Proof. Let A be the set of even-sized subsets of $[n]$ and let B be the set of odd-sized subsets of $[n]$. Consider the function

$$f(S) = \left\{ \begin{array}{ll} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{array} \right\}.$$

- $f : A \to B$ is a well defined function from A to B (why?).
- f^{-1} exists and equals f (why?) and has domain B (why?).

Therefore, f is a bijection, proving the statement, as desired.

Eyebrow-Raising Consequence: $\displaystyle\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0.$