Four Counting Questions (p. 2)

Here are four counting questions.

Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Q3. How many Pick-6 lottery tickets are there? (Choose six numbers between 1–40.)

Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Think Write Pair Share: Order these from smallest to largest.

_____ _____ _____ _____

Counting words

Definition: A **list** or **word** is an ordered sequence of objects.

Definition: A k-**list** or k-**word** is a list of length k.

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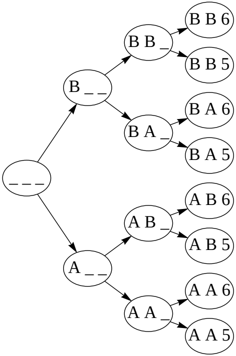

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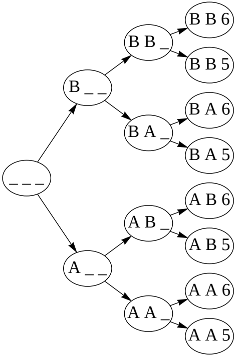

Alternatively: Notice two *independent* choices for each character. Multiply $2 \cdot 2 \cdot 2 = 8$.

The Product Principle

This illustrates:

The product principle: When counting lists $(l_1, l_2, \ldots, l_k)$,

IF there are c_1 choices for entry l_1, each leading to a different list,

AND IF there are c_i choices for entry l_i,
no matter the choices made for l_1 **through** l_{i-1},
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THEN there are $c_1 c_2 \cdots c_k$ such lists.

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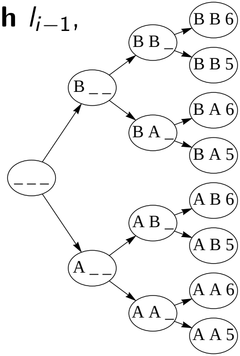

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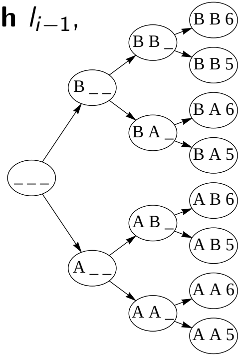

Caution: The product principle seems simple, but we must be careful when we use it.

Lists WITH repetition

Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?

Answer: Creating a word of length 8, with ____ choices for each character. Therefore, the number of 8-character passwords is ____.

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In general, the number of words of length k that can be made from an alphabet of length n and where repetition is allowed is n^k

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We can label the subsets by whether or not they contain s_i.

For example, for $n = 3$, we label the subsets $\left\{\begin{matrix} 000, 100, 010, 110, \\ 001, 101, 011, 111 \end{matrix}\right\}$

Permutations

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Answer: The number of choices for each lineup spot are:

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Question: How many 8-character passwords are there using A–Z, a–z, 0–9, containing no repeated character?

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In general, the number of words of length k that can be made from an alphabet of length n and where repetition is NOT allowed is $(n)_k$.

- That is, the number of k-permutations of an n-set is $(n)_k$.
- Special case: For n-permutations of an n-set: $n!$.

Notation

Some quantities appear frequently, so we use shorthand notation:

- $[n] := \{1, 2, \ldots, n\}$
- $2^S :=$ set of all subsets of S
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★ Leave answers to counting questions in terms of these quantities.

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Answer: $\binom{40}{6} = 3{,}838{,}380$.

- $\binom{n}{k}$ is called a **binomial coefficient**.
- Alternate phrasing: How many k-subsets of an n-set are there?
- The individual objects we are counting are unordered. They are subsets, not lists.

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Let's rearrange it. And prove it!

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Since we counted the same quantity twice, they must be equal!

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Answer:

How would you describe a k-multisubset of $[n]$?

Stars and Bars

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— is the same as —

Question: How many ways are there to place k indistinguishable balls into n distinguishable bins?

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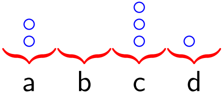

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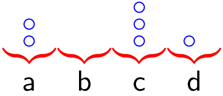

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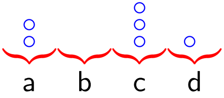

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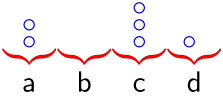

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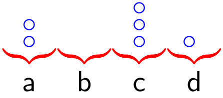

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$$\binom{k+n-1}{k} =: \left(\!\!\binom{n}{k}\!\!\right)$$

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Correct order:

Q2.	Order 9 baseball players $(9!)$	362,880
Q3.	Pick-6; numbers 1–40 $\binom{40}{6}$	3,838,380
Q4.	12 donuts from 30 $\left(\!\!\binom{30}{12}\!\!\right)$	7,898,654,920
Q1.	8-character passwords (62^8)	218,340,105,584,896

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	order matters (choose a list)	order doesn't matter (choose a set)
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repetition not allowed	$(n)_k$	$\binom{n}{k}$

Summary

	order matters (choose a list)	order doesn't matter (choose a set)
repetition allowed	n^k	$\left(\!\binom{n}{k}\!\right)$
repetition not allowed	$(n)_k$	$\binom{n}{k}$