Course Notes

Combinatorics, Fall 2013

Queens College, Math 636

Prof. Christopher Hanusa

On the web: `http://people.qc.cuny.edu/faculty/christopher.hanusa/courses/636fa13/`

Reference List

The following are books that I recommend to complement this course. I have asked that they be placed *on reserve* in the library.

Benjamin and Quinn. *Proofs that really count.*
Bóna. *A walk through combinatorics.*
Brualdi. *Introductory combinatorics.*
Graham, Knuth, and Patashnik. *Concrete mathematics.*
Mazur. *Combinatorics: A guided tour*
van Lint and Wilson. *A course in combinatorics.*

What is combinatorics?

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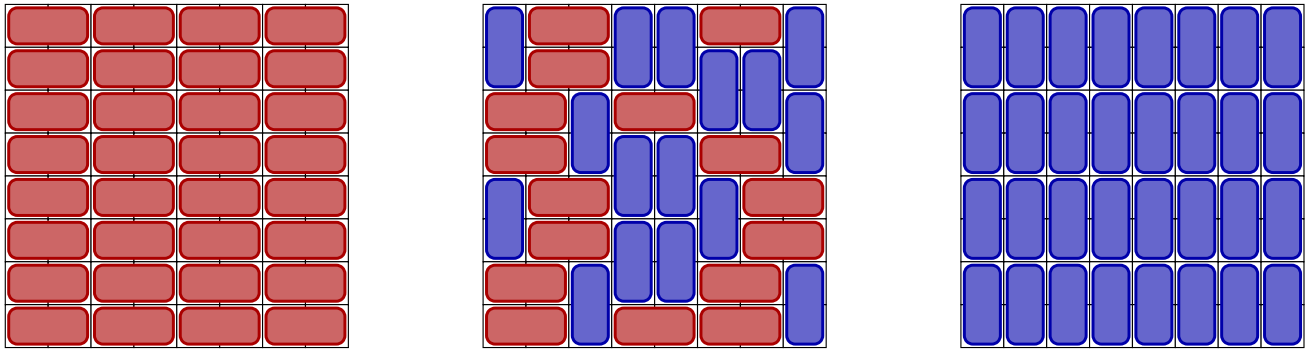

A **domino tiling** is a placement of dominoes on a region, where

- Each domino covers two squares.
- The dominoes cover the whole region and do not overlap.

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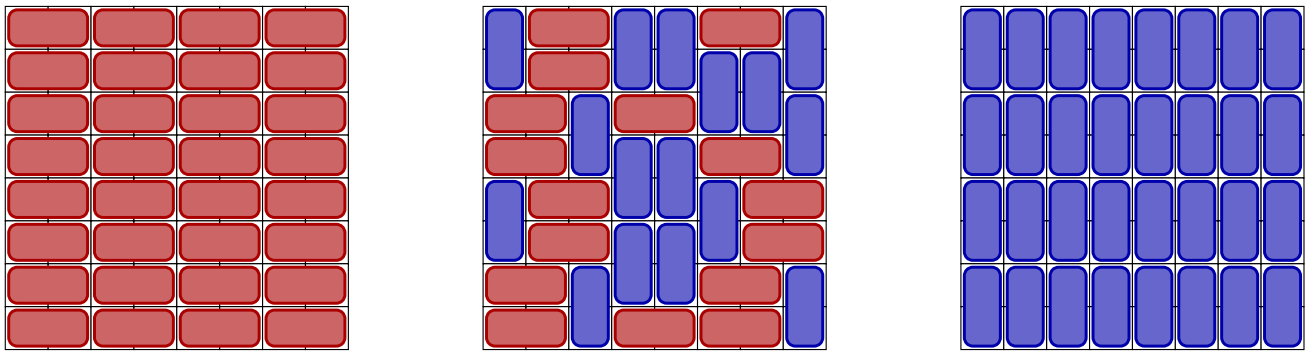

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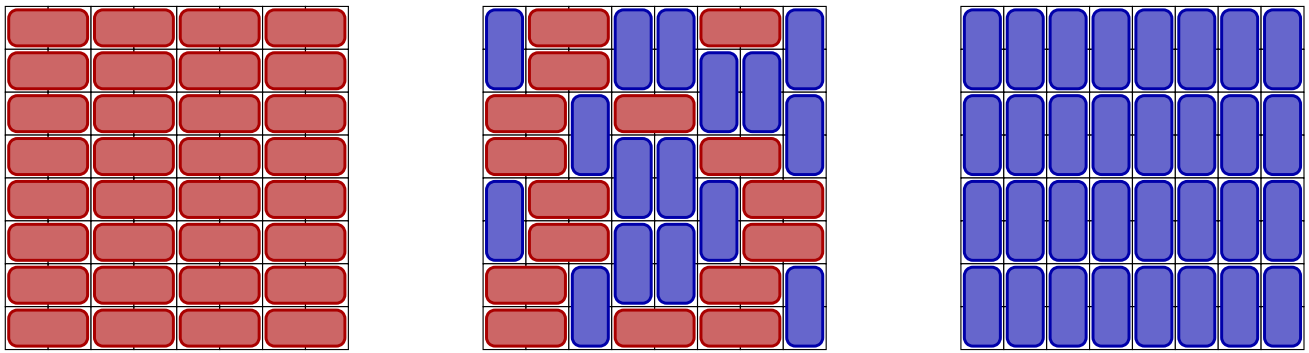

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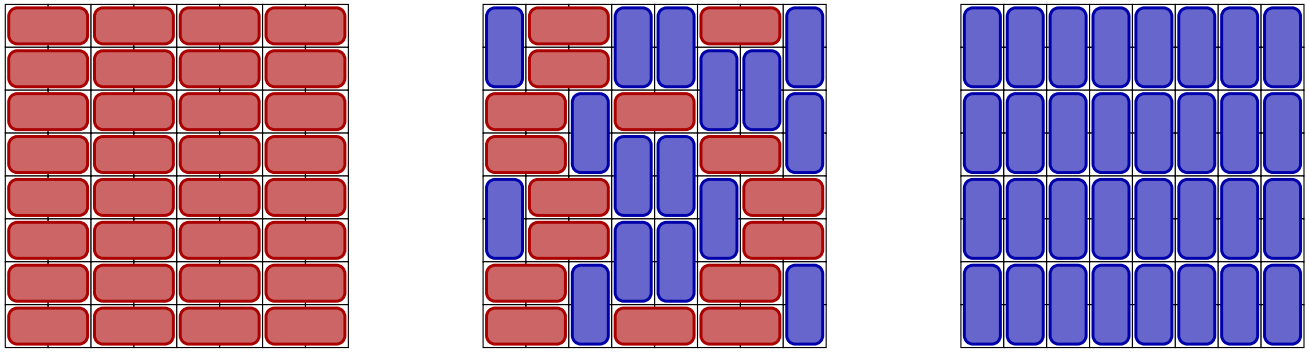

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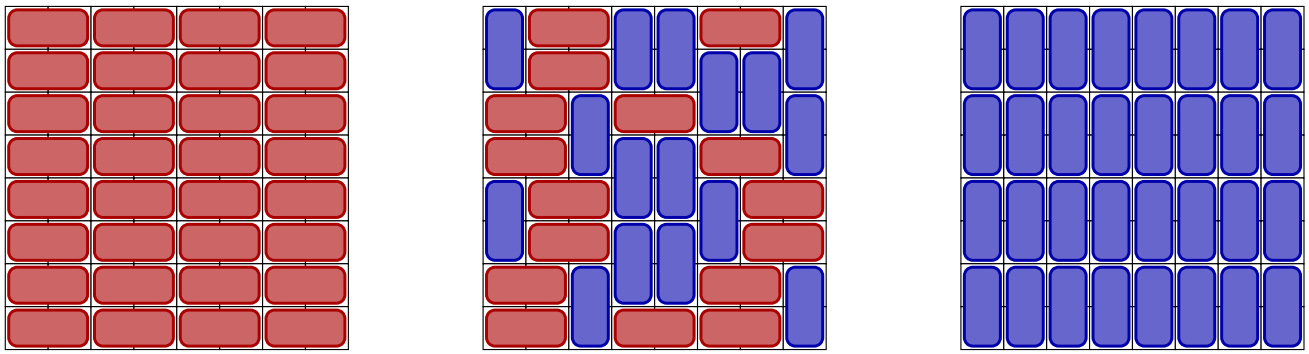

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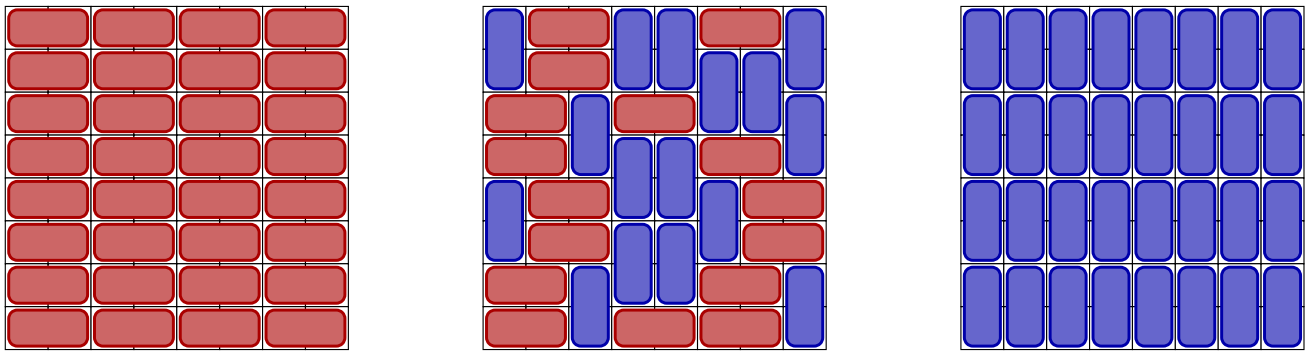

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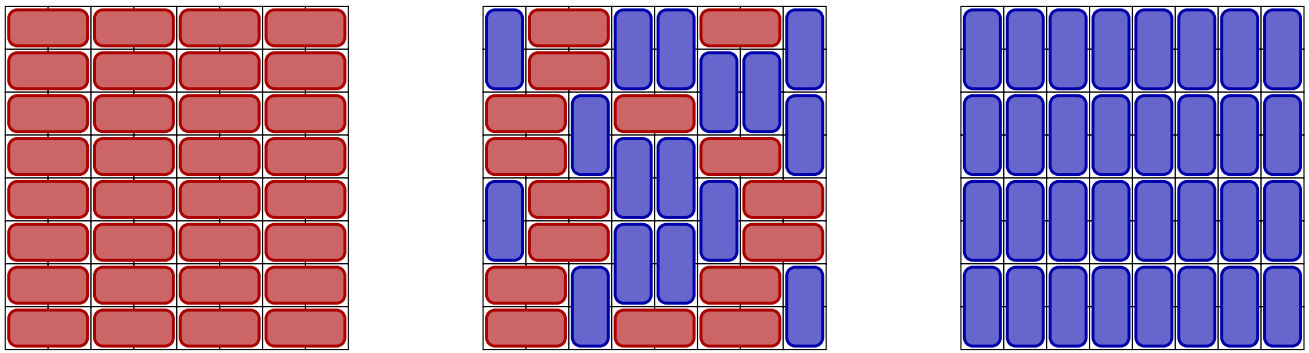

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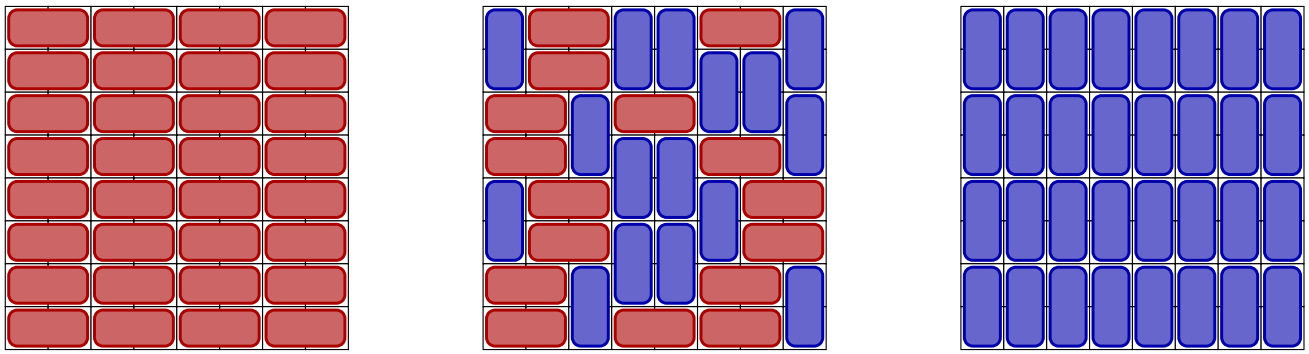

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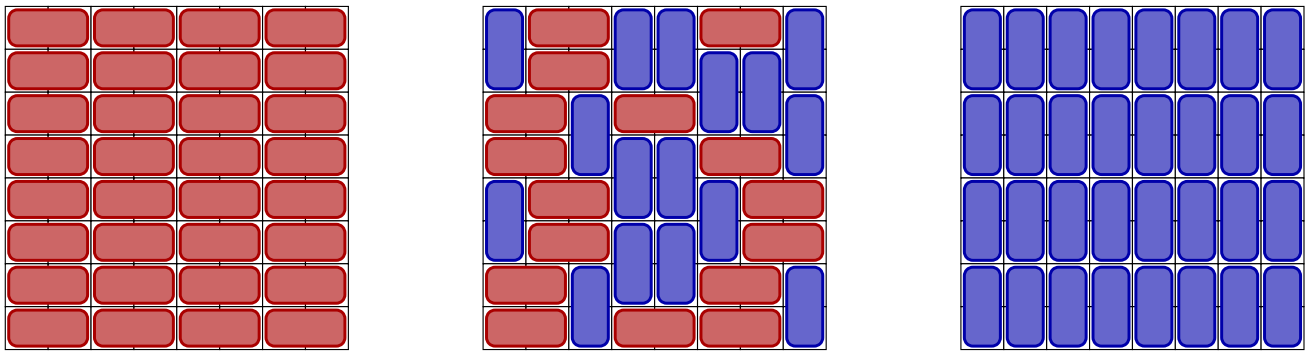

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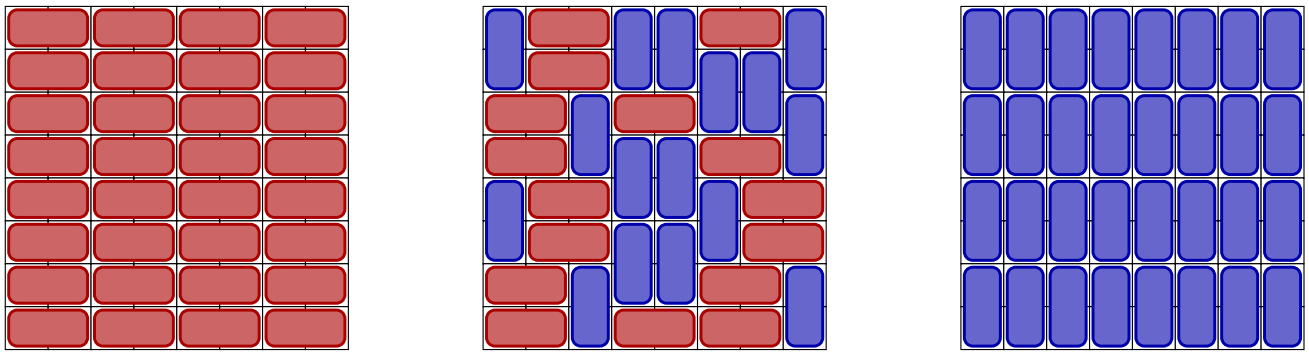

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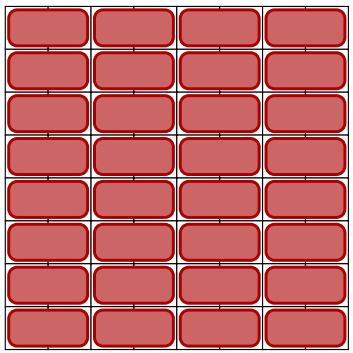
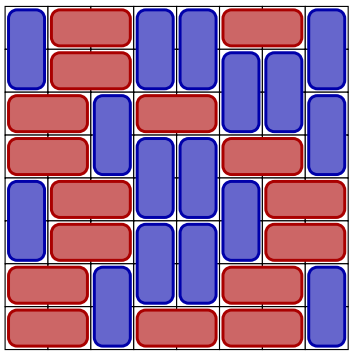
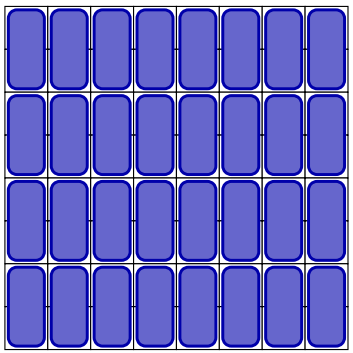

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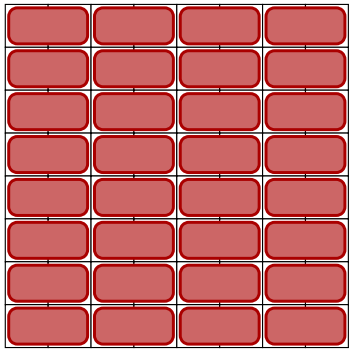
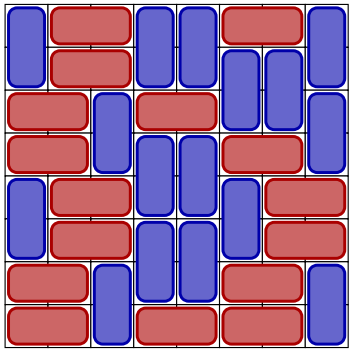
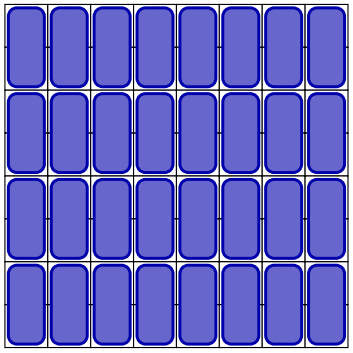

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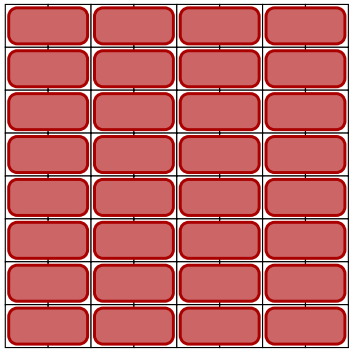
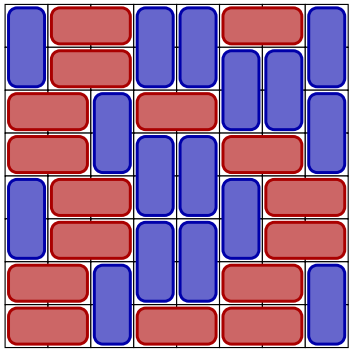
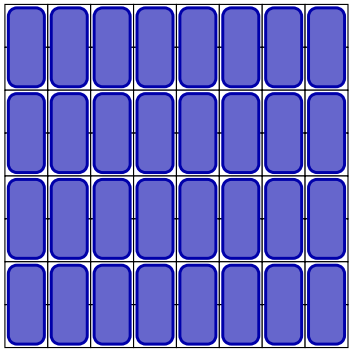

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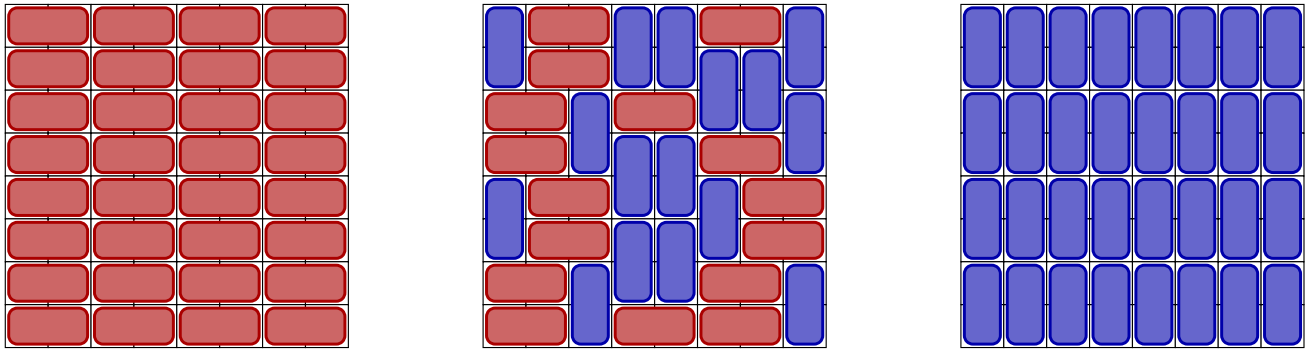

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We have the **answer**! What does it mean?

Domino tilings

How to determine the "answer"?

- Convert the chessboard into a combinatorial structure (a graph).
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Answer: If m and n are both even, then we have the **formula** (!):

$$\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right).$$

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(Requires many proofs.) (Uses a different kind of reasoning!)

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All homeworks online; first one due on Tuesday in two weeks.

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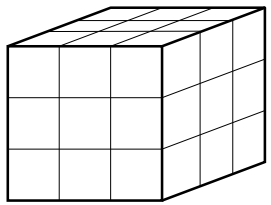

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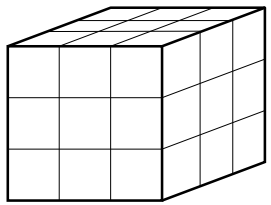

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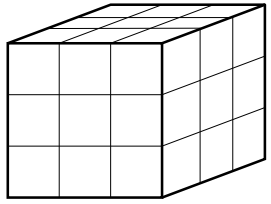

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Conjecture: 6 is the minimum possible number of cuts.
Proof:

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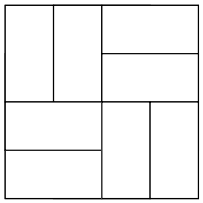

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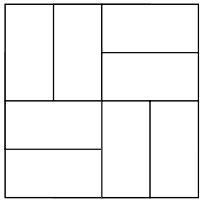

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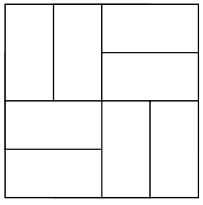

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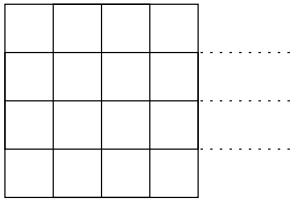

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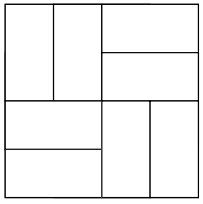

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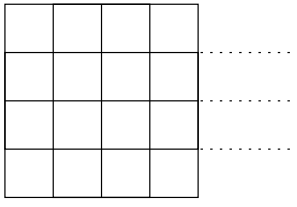

Every vertical domino must intersect exactly one of these separators; we can count the number of vertical dominoes by adding $x_1 + x_2 + x_3$.

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- However, a 4×4 chessboard can only hold 8 dominoes, a contradiction!

Therefore, it is impossible for a 4×4 domino tiling to have no fault lines.

Numbers are everywhere

Arrange yourselves into groups of four or six people,
With people you **don't know**.

- Introduce yourself. (your name, where you are from)
- What brought you to this class?
- Fill out **the front of** your notecard:
  - Write your name. (Stylize if you wish.)
  - Write a few words about yourself.
  - Draw something in the remaining space.
- Discuss with your groupmates why you wrote what you wrote.
- Exchange contact information. (phone / email / other)
- *Small talk suggestion:* Numbers are everywhere.
  What number do you identify yourself with and why?

Four Counting Questions (p. 2)

Here are four counting questions.

Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Q3. How many Pick-6 lottery tickets are there? (Choose six numbers between 1–40.)

Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Think Write Pair Share: Order these from smallest to largest.

_____ _____ _____ _____

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2. Alternating play, each player removes some number of counters from **either** pile.
3. The player who removes the last counter wins.

Let's play!

- First, get a feel for the game. Try starting with initial piles of (4,6), (5,5), (3,10), and (7,8).
- Next, start to develop some strategies for winning.
- Finally, determine conditions under which the first player will always win if she plays optimally, and similarly for the second player.

If you finish this before time is up, try playing Nim with three or more initial piles.