1. (a) (5 pts) Give a precise definition of a combinatorial statistic.

   (b) (5 pts) Let $\mathcal{T}_6$ be the set of all unlabeled trees on six vertices and let $t_6 = |\mathcal{T}_6|$. Determine t_6 by drawing every tree $T \in \mathcal{T}_6$ and verifying that there are no others.

   (c) (5 pts) We can define the combinatorial statistic $\Delta(T)$ to be the vertex of largest degree in a tree T. Write down the expression $\displaystyle\sum_{T\in\mathcal{T}_6} q^{\Delta(T)}$ and show that this is a q-analog of t_6.

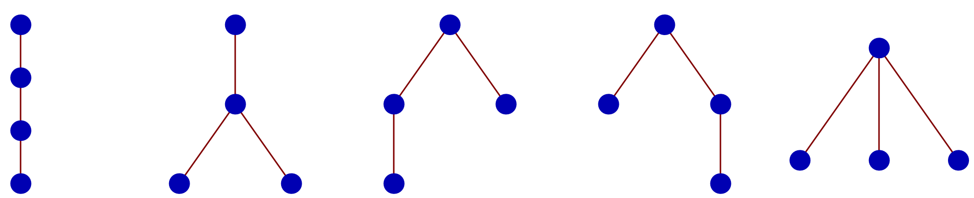
2. (10 pts) Let a, b, and c be nonzero real numbers. Find the coefficient of x^k in $\dfrac{a}{b+cx}$.

3. (10 pts) Give the generating function for partitions of n where all parts that appear occur **at least two** times. Write your answer as an infinite product using $\prod$ notation.
   [*For example,* $\lambda = 6\,6\,6\,3\,3\,1\,1$ *is valid but* $\lambda = 5\,5\,3\,2\,2\,2$ *is not because* 3 *occurs only once.*]

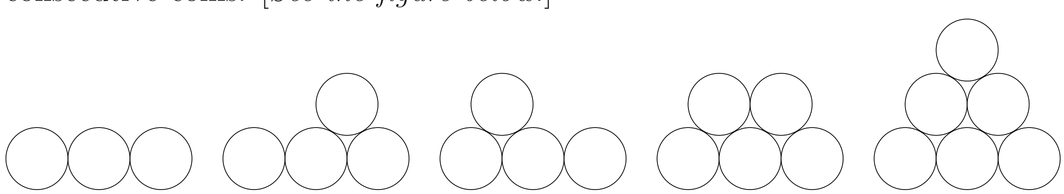
4. (10 pts) You have infinitely many squares of colors $\{\text{red}, \text{blue}, \text{green}\}$ and dominoes of colors $\{\text{orange}, \text{yellow}, \text{indigo}, \text{purple}\}$. Let h_n be the number of ways in which we can cover an $n \times 1$ board with these colorful tiles. Find a linear recurrence that $\{h_n\}$ satisfies (including initial conditions) and give the generating function $h(x) = \sum_{n\geq 0} h_n x^n$.

5. (10 pts) Below are **TWO** combinatorial interpretations for the Catalan number C_n. Choose **ONE** of these combinatorial interpretations and **prove** that it is indeed counted by the Catalan numbers.

   (a) The number of rooted trees with n vertices. [*See the figure below.*]
   [*Note: Unlike the binary trees we discussed in class, we allow every vertex to have* **arbitrarily** *many children—there is no such thing as a "left child" nor "right child".*]



   (b) The number of ways to stack coins in the plane where the bottom row consists of n consecutive coins. [*See the figure below.*]



[**Bonus:** *Prove that the* other *combinatorial interpretation is also counted by the Catalan numbers.*]