q-analogs

Definition: A q-**analog** of a number c is an expression $f(q)$ such that $\lim_{q\to 1} f(q) = c$.

Example: $\dfrac{1-q^n}{1-q} = \left(1 + q + q^2 + \cdots + q^{n-2} + q^{n-1}\right)$ is a q-analog of n because $\lim_{q\to 1} \frac{1-q^n}{1-q} = n$.

We write $[n]_q = \frac{1-q^n}{1-q}$.

q-analogs work hand in hand with combinatorial statistics.

If stat is a combinatorial statistic on a set S ($\mathrm{stat} : S \mapsto \mathbb{N}$), then $\sum_{s\in S} q^{\mathrm{stat}(s)}$ is a q-analog of $|S|$ because

$$\lim_{q\to 1} \sum_{s\in S} q^{\mathrm{stat}(s)} = \sum_{s\in S} 1^{\mathrm{stat}(s)} = \sum_{s\in S} 1 = |S|$$

Inversion statistics

Given a perm. $\pi = \pi_1\pi_2\cdots\pi_n$, the inversion number $\mathrm{inv}(\pi) = |\{i < j : \pi(i) > \pi(j)\}|$.

$n \backslash i$	0	1	2	3	4	5	6
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1

Question: What is the generating function $\sum_{\pi\in S_n} q^{\mathrm{inv}(\pi)}$?

n	$\sum_{\pi\in S_n} q^{\mathrm{inv}(\pi)}$	
1	$1q^0$	$= 1$
2	$1q^0 + 1q^1$	$= (1+q)$
3	$1q^0 + 2q^1 + 2q^2 + 1q^3$	$= (1+q+q^2)(1+q)$
4	$1q^0 + 3q^1 + 5q^2 + 6q^3 + 5q^4 + 3q^5 + 1q^6$	$=$

Conjecture: $\sum_{\pi\in S_n} q^{\mathrm{inv}(\pi)} = [n]_q \cdots [1]_q =: [n]_q!$, the q-**factorial**.

(Note: $q = 1$ works!)

Inversion Statistics

Theorem: $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = [n]_q!$

Proof. There exists a bijection

$$\left\{ \begin{array}{c} \text{permutations} \\ \pi \in S_n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{lists } (a_1, \ldots, a_n) \\ \text{where } 0 \le a_i \le n - i \end{array} \right\}.$$

Given a permutation π, create its **inversion table**. Define a_i to be the number of entries j to the left of i that are smaller than i. Then $\mathrm{inv}(\pi) = a_1 + a_2 + \cdots + a_n$.

Example. The inversion table of $\pi = 4\,3\,1\,5\,2$ is $(3, 2, 0, 1, 0)$.

$$\begin{aligned} \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} &= \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \cdots \sum_{a_n=0}^{0} q^{a_1+a_2+\cdots+a_n} \\ &= \left(\sum_{a_1=0}^{n-1} q^{a_1} \right) \left(\sum_{a_2=0}^{n-2} q^{a_2} \right) \cdots \left(\sum_{a_n=0}^{0} q^{a_n} \right) \\ &= \quad [n]_q \quad [n-1]_q \quad \cdots \quad [1]_q \quad = \quad [n]_q! \end{aligned}$$

Notes

- The statement that the inversion and major statistics have the same distribution corresponds to the equation

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)}.$$

- Now that we have a q-analog of factorials, we can define a q-analog of binomial coefficients. Define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

  These polynomials are called the q-**binomial coefficients** or **Gaussian polynomials**.
  - $\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$.
  - They are indeed polynomials.
  - *Example.* $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$
- Combinatorial interpretations of q-binomial coefficients!

Combinatorial interpretations of q-binomial coefficients

Consider set $S_{k,n-k}$ of permutations of the multiset $\{1^k, 2^{n-k}\}$.
Define $\text{inv}(\pi) = |\{i < j : \pi(i) > \pi(j)\}|$.

Example. $\pi = 1\,1\,2\,2\,1\,2\,1\,1\,2\,2$ is a permutation of $\{1^5, 2^5\}$.
Then $\text{inv}(\pi) = 0 + 0 + 3 + 3 + 0 + 2 + 0 + 0 + 0 + 0 = 8$.

Then $\sum_{\pi \in S_{k,n-k}} q^{\text{inv}(\pi)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$. (Note $|S_{k,n-k}| = \binom{n}{k}$.)

This is a refinement of these permutations in terms of inversions.

Consider the set $\mathcal{P}$ of lattice paths from $(0,0)$ to (a,b).
Let $\text{area}(P)$ be the area above a path P.
Then $\sum_{P \in \mathcal{P}} q^{\text{area}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$. (Note $|\mathcal{P}| = \binom{a+b}{a}$.)

This can also be used to give a q-analog of the Catalan numbers.

Multivariate generating functions