Combinatorial statistics

Given a set of combinatorial objects $\mathcal{A}$, a **combinatorial statistic** is an integer given to every element of the set.

In other words, it is a function $\mathcal{A} \to \mathbb{Z}_{\geq 0}$.

Example. Let $\mathcal{S}$ be the set of subsets of $\{1, 2, 3\}$.
The cardinality of a set is a combinatorial statistic on $\mathcal{S}$.

$$\big|\emptyset\big| = 0 \quad \big|\{1\}\big| = 1 \quad \big|\{2\}\big| = 1 \quad \big|\{3\}\big| = 1$$
$$\big|\{1,2\}\big| = 2 \quad \big|\{1,3\}\big| = 2 \quad \big|\{2,3\}\big| = 2 \quad \big|\{1,2,3\}\big| = 3$$

Combinatorial statistics provide a *refinement* of counting.

less information ⟵ ⟶ *more information*

counting

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statistics

0	1	2	3
1	3	3	1

complete enumeration

$\emptyset$ $\{1\}$ $\{2\}$ $\{3\}$

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,2,3\}$

More statistics

Questions involving combinatorial statistics:

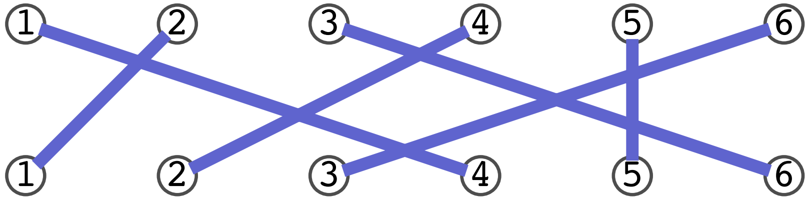
- What is the *distribution* of the statistics?
- What is the *average size* of an object in the set?
- Which statistics have the same distribution?
  - Insight into their structure.
  - Provides non-trivial bijections in the set?

A especially rich playground involves *permutation statistics*.

Representations of permutations

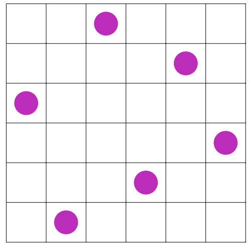
One-line notation: $\pi = 4\,1\,6\,2\,5\,3$ Cycle notation: $\pi = (1\,4\,2)(3\,6)(5)$

String diagram:



(only two crossings at a time)

Matrix-like diagram:

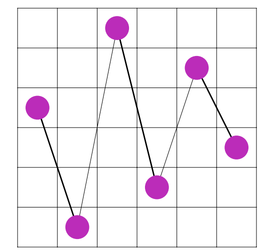


Descent statistic

Definition: Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation.

A **descent** is a position i such that $\pi_i > \pi_{i+1}$.

Define $\mathrm{des}(\pi)$ to be the **number of descents** in π.



Example. When $\pi = 416253$, $\mathrm{des}(\pi) = 3$ since $4 \searrow 1$, $6 \searrow 2$, $5 \searrow 3$.

Question: How many n-permutations have d descents?

$\mathrm{des}(12) = 0$ $\quad$ $\mathrm{des}(123) = \_$ $\quad$ $\mathrm{des}(213) = \_$ $\quad$ $\mathrm{des}(312) = \_$

$\mathrm{des}(21) = 1$ $\quad$ $\mathrm{des}(132) = \_$ $\quad$ $\mathrm{des}(231) = \_$ $\quad$ $\mathrm{des}(321) = \_$

$n \backslash d$	0	1	2	3	4
1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
5	1	26	66	26	1

What are the possible values for $\mathrm{des}(\pi)$?

Note the symmetry. If π has d descents, its reverse $\hat{\pi}$ has ___ descents.

These are the **Eulerian numbers**.

Eulerian Numbers

Definition: $A_{n,k}$ = number of n-permutations with $k-1$ descents.

Theorem: $A_{n,k+1} = (k+1)A_{n-1,k+1} + (n-k)A_{n-1,k}$

Proof. Ask: How many n-permutations have k descents?

LHS: $A_{n,k+1}$, of course!

RHS: Insert the number n into an $(n-1)$-permutation.
When n is inserted into an $(n-1)$-permutation with d descents, the resulting n-permutation either has

- d descents (If n inserted in a position that is a descent or at end.)
- $d+1$ descents (If n inserted in a position that is not a descent.)

Conclusion: An n-perm with k descents can arise by inserting n:

- into a perm with k existing descents in $(k+1)A_{n-1,k+1}$ ways.
- into a perm with $k-1$ existing descents in $(n-k)A_{n-1,k}$ ways.

Eulerian Numbers

The initial conditions
$A_{n,1} = 1$ and $A_{n,n} = 1$ for all n
along with the recurrence
$A_{n,k+1} =$
$(k+1)A_{n-1,k+1} + (n-k)A_{n-1,k}$
allow us to fill the chart:

n	$A_{n,1}$	$A_{n,2}$	$A_{n,3}$	$A_{n,4}$	$A_{n,5}$	$A_{n,6}$
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	**1**	**26**	66	26	1	
6	1	**57**				1

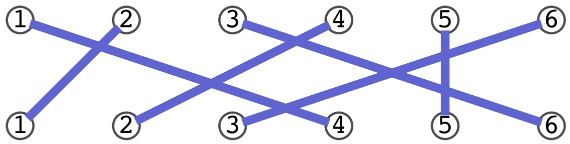
Fact: The Eulerian numbers satisfy the following identities.

$$A_{n,k} = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n.$$

$$S(n,r) = \frac{1}{r!} \sum_{k=0}^{r} A_{n,k} \binom{n-k}{r-k}$$

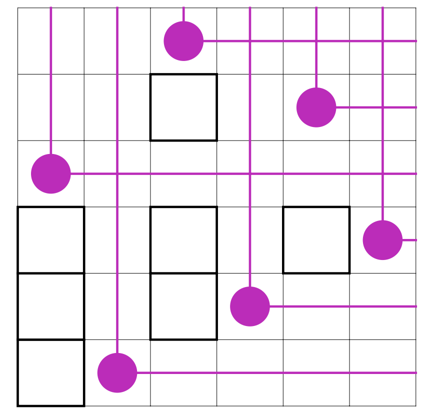
Inversion statistic

Definition: Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. An **inversion** is a pair $i < j$ such that $\pi_i > \pi_j$.



Define $\text{inv}(\pi)$ as the **number of inversions** in π.

Example. When $\pi = 416253$, $\text{inv}(\pi) = 7$ since $4>1$, $4>2$, $4>3$, $6>2$, $6>5$, $6>3$, $5>3$.
In a string diagram $\text{inv}(\pi) =$ number of crossings.
In a matrix diagram $\text{inv}(\pi)$, draw *Rothe diagram*:



$\text{inv}(12) = 0$	$\text{inv}(123) = \_$	$\text{inv}(213) = \_$	$\text{inv}(312) = \_$
$\text{inv}(21) = 1$	$\text{inv}(132) = \_$	$\text{inv}(231) = \_$	$\text{inv}(321) = \_$

$n\backslash i$	0	1	2	3	4	5	6
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1

What are the possible values for $\text{inv}(\pi)$?

The inversion number is a good way to count how "far away" a permutation is from the identity.

Gaussian polynomials

Definition: $b_{n,k}$ = number of n-permutations with k inversions.

Theorem: Let $k \leq n$. Then $b_{n+1,k} = b_{n+1,k-1} + b_{n,k}$

Proof. Ask: How many $(n+1)$-permutations have k descents?

LHS: $b_{n+1,k}$, evidently!

RHS: Condition on the position of $(n+1)$.
The $(n+1)$-perms with k descents and $(n+1)$ in the last position are in bijection with ____________________, and are counted by ___.

If $(n+1)$ is not in the last position, switch it with its right neighbor. We recover an $(n+1)$-permutation with $k-1$ descents with the added condition that ____________________________.

Since $k \leq n$, then every $(n+1)$-permutation with $k-1$ inversions satisfy this condition, (WHY?)

We conclude that there are $b_{n+1,k-1}$ ways in which this can happen.

Major index

Definition: Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation.

Define maj(π), the **major index** of π, to be sum of the descents of π.

[Named after Major Percy MacMahon. (British army, early 1900's)]

Example. When $\pi = 416253$, maj(π) = 9 since the descents of π are in positions 1, 3, and 5.

maj(12) = 0 maj(123) = _ maj(213) = _ maj(312) = _

maj(21) = 1 maj(132) = _ maj(231) = _ maj(321) = _

$n\backslash m$	0	1	2	3	4	5	6
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1

What are the possible values for maj(π)?

The distribution of maj(π) IS THE SAME AS the distribution of inv(π)!

A statistic that has the same distribution as inv is called **Mahonian**.

There's always more to learn!!!

Theorem: inv and maj are equidistributed on S_n.

Proofs exist using generating functions and using bijections.

- Find a bijection $f : S_n \to S_n$ such that $\mathrm{maj}(\pi) = \mathrm{inv}(f(\pi))$.

References :

Miklós Bóna. Combinatorics of Permutations, CRC, 2004.

T. Kyle Petersen. Two-sided Eulerian numbers via balls in boxes. http://arxiv.org/abs/1209.6273

The Combinatorial Statistic Finder. http://findstat.org/