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- There is an amazingly nice formula for the number of labeled trees.

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Thm 6.2.5 The number of labeled trees is . (drumroll....)

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Thm 6.2.5 The number of labeled trees is n^{n-2}. (drumroll....)

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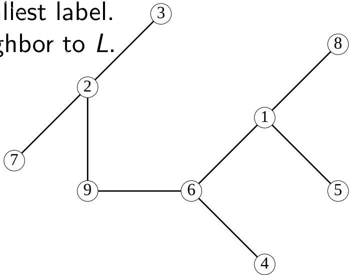

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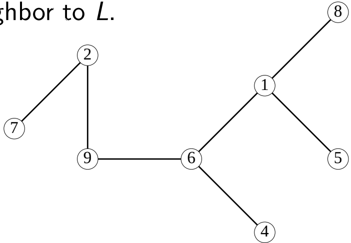

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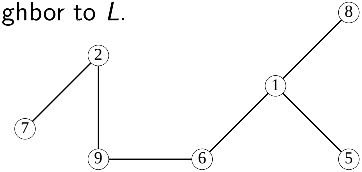

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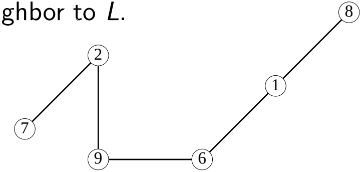

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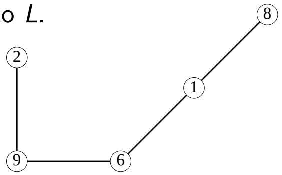

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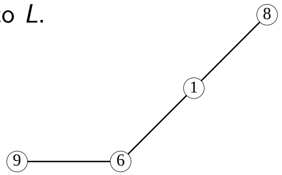

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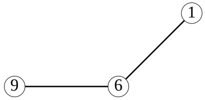

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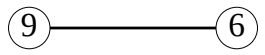

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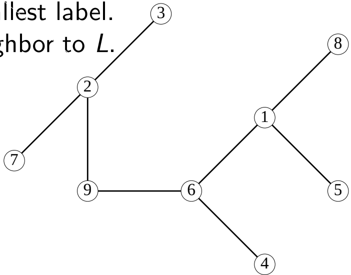

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★ This rule is well defined. ★

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Conjecture: The number of binary trees on n vertices is ______.

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