Catalan Numbers

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On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

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triangulations of an $(n+2)$-gon	lattice paths from $(0,0)$ to (n,n) above $y=x$
sequences with n $+1$'s, n -1's with positive partial sums	multiplication schemes to multiply $n+1$ numbers

Catalan Number Interpretations

When $n = 3$, there are $c_3 = 5$ members of these families of objects:

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4. Ways to multiply $n+1$ numbers together two at a time.

Catalan Bijections

We claim that these objects are all counted by the Catalan numbers. So there should be bijections between the sets!

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Bijection 1: $\boxed{\begin{array}{c}\text{triangulations}\\ \text{of an } (n+2)\text{-gon}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{multiplication schemes}\\ \text{to multiply } n+1 \text{ numbers}\end{array}}$

Rule: Label all but one side of the $(n+2)$-gon in order. Work your way in from the outside to label the interior edges of the triangulation: When you know two sides of a triangle, the third edge is the product of the two others. Determine the mult. scheme on the last edge.

Catalan Bijections

Bijection 2: $\boxed{\text{multiplication schemes to multiply } n+1 \text{ \#s}} \longleftrightarrow \boxed{\text{seqs with } n \; +1\text{'s, } n \; -1\text{'s with positive partial sums}}$

Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by $+1$'s and the parentheses by -1's.

Catalan Bijections

Bijection 3: $\boxed{\text{seqs with } n\ {+1}\text{'s, } n\ {-1}\text{'s with positive partial sums}} \longleftrightarrow \boxed{\text{lattice paths } (0,0) \text{ to } (n,n) \text{ above } y = x}$

A sequence of $+$'s and $-$'s converts to a sequence of N's and E's, which is a path in the lattice.

Catalan Numbers

The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the generating function equation that $C(x)$ satisfies:

$$C(x) = 1 + xC(x)^2.$$

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Example. $\boxed{\text{triangulations of an } (n+2)\text{-gon}}$

Here, x represents one side of the polygon

Either the triangulation has a side or not.

1. No side: Empty triangulation: x^0.
2. Every other triangulation has one side (x contribution) and breaks down as two other triangulations $C(x)^2$.

Catalan Numbers

Example. $\boxed{\text{lattice paths } (0,0) \text{ to } (n,n) \text{ above } y = x}$

Here, x represents an up-step down-step pair.

Either the lattice path starts with a vertical step or not.

1. No step: Empty lattice path: x^0.
2. Every other lattice path has one vertical step up from diag. and a first horizontal step returning to diag. (x contribution). Between these steps, after these steps are two lattice paths $C(x)^2$.

Therefore, $C(x) = 1 + xC(x)^2$.

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.

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Therefore, $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

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