# Catalan Numbers

| $c_0$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ | $c_7$ | $c_8$ | $c_9$ | $c_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4862 | 16796 |

On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Richard Stanley has compiled a list of combinatorial interpretations of Catalan numbers. As of 7/12, numbered (a) to (z), ... ($a^8$) to ($t^8$).

| | |
|---|---|
| triangulations of an $(n+2)$-gon | lattice paths from $(0, 0)$ to $(n, n)$ above $y = x$ |
| sequences with $n$ +1's, $n$ −1's with positive partial sums | multiplication schemes to multiply $n + 1$ numbers |

# Catalan Number Interpretations

When $n = 3$, there are $c_3 = 5$ members of these families of objects:

1. Triangulations of an $(n+2)$-gon

2. Lattice paths from $(0,0)$ to $(n,n)$ staying above $y = x$

3. Sequences of length $2n$ with $n$ $+1$'s and $n$ $-1$'s such that every partial sum is $\geq 0$

4. Ways to multiply $n+1$ numbers together two at a time.

# Catalan Bijections

We claim that these objects are all counted by the Catalan numbers. So there should be bijections between the sets!

Bijection 1: | triangulations of an $(n+2)$-gon | $\longleftrightarrow$ | multiplication schemes to multiply $n+1$ numbers |

Rule: Label all but one side of the $(n+2)$-gon in order. Work your way in from the outside to label the interior edges of the triangulation: When you know two sides of a triangle, the third edge is the product of the two others. Determine the mult. scheme on the last edge.

# Catalan Bijections

Bijection 2: | multiplication schemes to multiply $n+1$ #s | $\longleftrightarrow$ | seqs with $n$ $+1$'s, $n$ $-1$'s with positive partial sums |

Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by $+1$'s and the parentheses by $-1$'s.

# Catalan Bijections

Bijection 3: | seqs with $n$ $+1$'s, $n$ $-1$'s with positive partial sums | $\longleftrightarrow$ | lattice paths $(0,0)$ to $(n,n)$ above $y = x$ |

A sequence of $+$'s and $-$'s converts to a sequence of $N$'s and $E$'s, which is a path in the lattice.

# Catalan Numbers

The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the generating function equation that $C(x)$ satisfies:

$$C(x) = 1 + xC(x)^2.$$

Example. | triangulations of an $(n+2)$-gon |

Here, $x$ represents one side of the polygon

Either the triangulation has a side or not.

1. No side: Empty triangulation: $x^0$.
2. Every other triangulation has one side ($x$ contribution) and breaks down as two other triangulations $C(x)^2$.

# Catalan Numbers

Example.

lattice paths $(0,0)$ to $(n,n)$ above $y = x$

Here, $x$ represents an up-step down-step pair.

Either the lattice path starts with a vertical step or not.

1. No step: Empty lattice path: $x^0$.
2. Every other lattice path has one vertical step up from diag. and a first horizontal step returning to diag. ($x$ contribution). Between these steps, after these steps are two lattice paths $C(x)^2$.

Therefore, $C(x) = 1 + xC(x)^2$.

# A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.
Do we take the positive or negative root? Check $x = 0$.

Now extract coefficients to prove the formula for $c_n$.

Claim: $\sqrt{1-4x} = 1 + \sum_{k\geq 1} \frac{-2}{k}\binom{2(k-1)}{k-1}x^k.$ (Next slide.)

Conclusion.
$$\begin{aligned}\tfrac{1}{2x}(1-\sqrt{1-4x}) &= -\tfrac{1}{2x}\sum_{k\geq 1}\tfrac{-2}{k}\tbinom{2(k-1)}{k-1}x^k \\ &= \sum_{k\geq 1}\tfrac{1}{k}\tbinom{2(k-1)}{k-1}x^{k-1} \\ &= \sum_{n\geq 0}\tfrac{1}{n+1}\tbinom{2n}{n}x^n\end{aligned}$$

Therefore, $c_n = \frac{1}{n+1}\binom{2n}{n}$.

# Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

$$\sqrt{1-4x} = ((-4x)+1)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k \quad \text{Expand } \binom{1/2}{k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k \quad \text{Denom. of } \tfrac{1}{2}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k \quad \text{Factor } -2\text{'s}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k 4^k x^k \quad \text{Simplify; rewrite prod.}$$

$$= 1 + \sum_{k=1}^{\infty} -\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-3)\cdot(2k-2)}{k!\cdot 2\cdot 4\cdots(2k-2)} 2^k x^k \quad \text{Write as factorials}$$

$$= 1 + \sum_{k=1}^{\infty} -\frac{(2k-2)!}{k!(2^{k-1})1\cdot 2\cdots(k-1)} 2^k x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{-2}{k}\frac{(2k-2)!}{(k-1)!(k-1)!} x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{-2}{k}\binom{2(k-1)}{k-1} x^k$$