Tiling a board with dominos and squares

Question: How many ways are there to tile a $1 \times n$ board using only dominoes and squares?

Definition: Let $f_n = \#$ of ways to tile a $2 \times n$ board.

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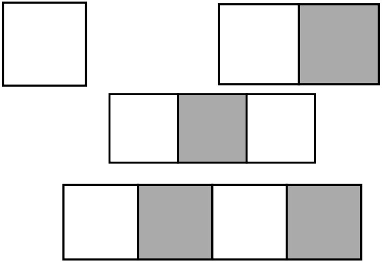

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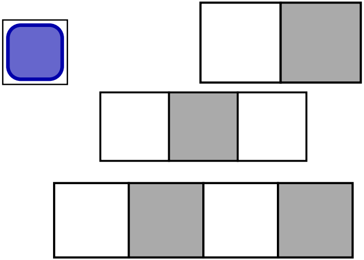

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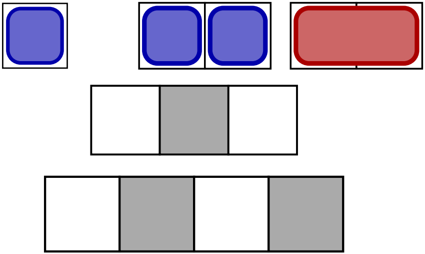

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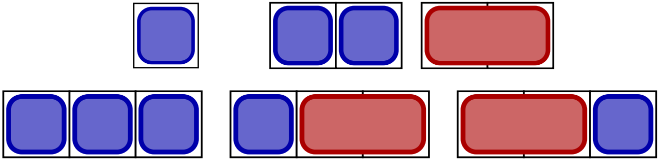

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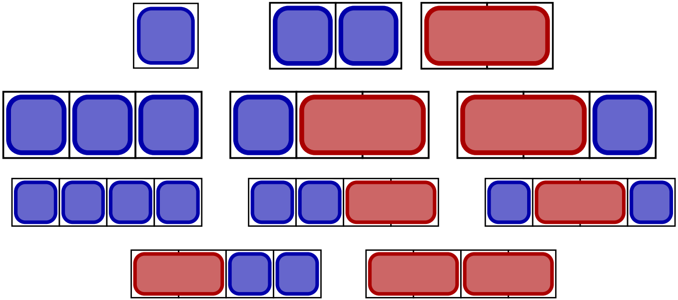

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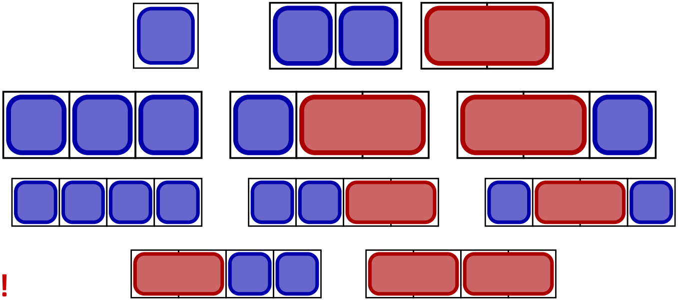

Fibonacci!

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Fibonacci numbers f_n satisfy

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1	2	**3**	**5**	8	13	21	**34**	55	89	144	233	377	610

$$f_8 = f_4^2 + f_3^2$$

$$34 = 25 + 9$$

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$$f_{14} = f_7^2 + f_6^2$$

$$610 = 441 + 169$$

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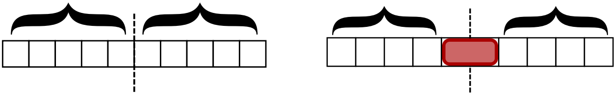
Either there is... Or there isn't...



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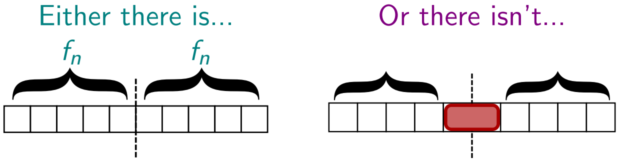

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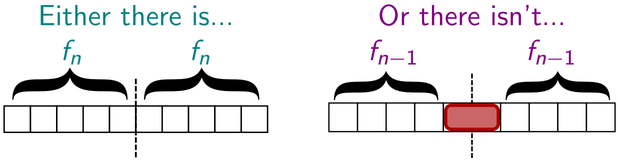

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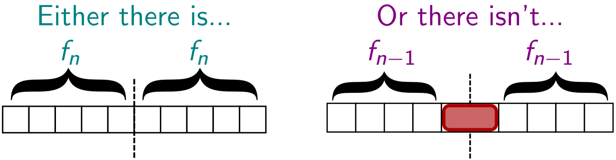

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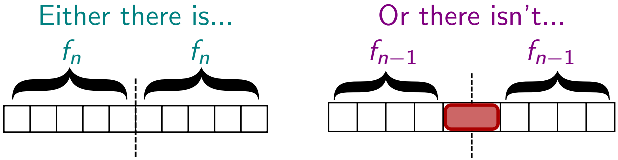

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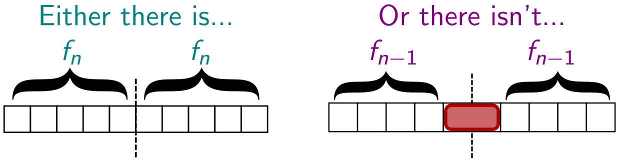

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Further reading:

Arthur T. Benjamin and Jennifer J. Quinn
Proofs that Really Count, MAA Press, 2003.