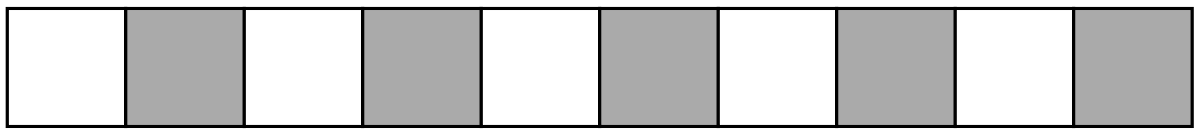
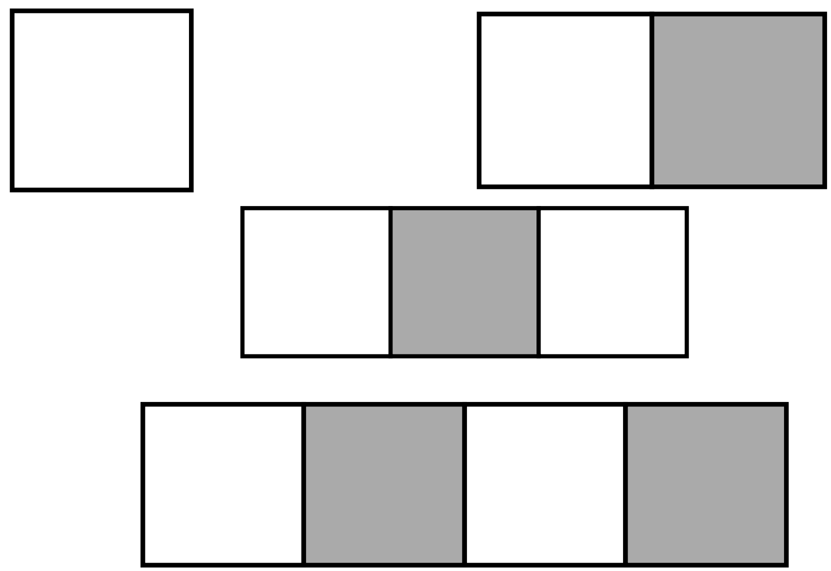
Tiling a board with dominos and squares

Question: How many ways are there to tile a $1 \times n$ board using only dominoes and squares?



Definition: Let $f_n = \#$ of ways to tile a $2 \times n$ board.

$f_0 = 1$
$f_1 =$
$f_2 =$
$f_3 =$
$f_4 =$



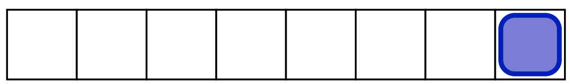
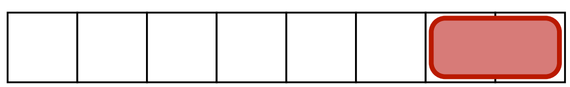
Why Fibonacci?

Fibonacci numbers f_n satisfy

- $f_0 = f_1 = 1$ ✓
- $f_n = f_{n-1} + f_{n-2}$ ✓

There are f_n tilings of a $1 \times n$ board

Every tiling ends in either:

- a square 
  - **How many?** Fill the initial $1 \times (n-1)$ board in f_{n-1} ways.
- a domino 
  - **How many?** Fill the initial $1 \times (n-2)$ board in f_{n-2} ways.

Total: $f_{n-1} + f_{n-2}$

Fibonacci identities

We have a new definition for Fibonacci:

$$f_n = \text{the number of square-domino tilings of a } 1 \times n \text{ board.}$$

This *combinatorial interpretation* of the Fibonacci numbers provides a framework to prove identities.

- Did you know that $f_{2n} = (f_n)^2 + (f_{n-1})^2$?

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}
1	2	3	5	8	**13**	**21**	34	55	89	144	233	377	**610**

$$f_{14} = f_7^2 + f_6^2$$

$$610 = 441 + 169$$

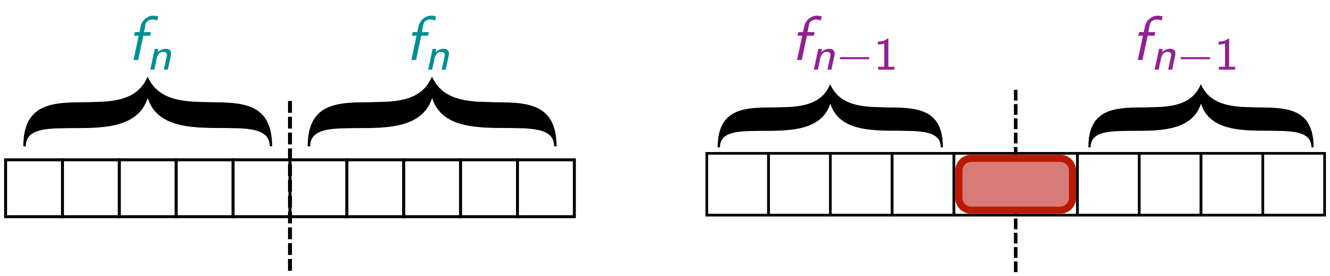
Proof that $f_{2n} = (f_n)^2 + (f_{n-1})^2$

Proof. How many ways are there to tile a $1 \times (2n)$ board?

Answer 1. Duh, f_{2n}.

Answer 2. Ask whether there is a break in the middle of the tiling:

Either there is... Or there isn't...



For a total of $(f_n)^2 + (f_{n-1})^2$ tilings.

We counted f_{2n} in two different ways, so they must be equal. $\square$

Further reading:

Arthur T. Benjamin and Jennifer J. Quinn
Proofs that Really Count, MAA Press, 2003.