Multiplying two generating functions

Let $A(x) = \sum_{k \geq 0} a_k x^k$ and $B(x) = \sum_{k \geq 0} b_k x^k$.
What is the coefficient of x^k in $A(x)B(x)$?

When expanding the product $A(x)B(x)$ we multiply terms $a_i x^i$ in A by terms $b_j x^j$ in B. This product contributes to the coefficient of x^k in AB only when ________________________________.

Therefore, $A(x)B(x) = \sum_{k \geq 0} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k$

An interpretation of this theorem:
If a_k counts all sets of size k of type "S", and b_k counts all sets of size k of type "T", then $[x^k]\big(A(x)B(x)\big)$ counts all pairs of sets (S, T) where the total number of elements in both sets is k.

Multiplying two generating functions

Example. What is the coefficient of x^7 in $\frac{x^3(1+x)^4}{(1-2x)}$?

A special case of the above: $(A(x))^2 = \sum_{k\geq 0} \left(\sum_{i_1+i_2=k} a_{i_1} a_{i_2} \right) x^k$

Similar to above, the coefficient of x^k of $(A(x))^n$ is:

$$(A(x))^n = \sum_{k\geq 0} \left(\sum_{i_1+i_2+\cdots+i_n=k} a_{i_1} a_{i_2} \cdots a_{i_n} \right) x^k$$

Compositions of Generating Functions

Question: Let $F(x) = \sum_{n\geq 0} f_n x^n$ and $G(x) = \sum_{n\geq 0} g_n x^n$.
What does $H(x) = F\big(G(x)\big)$ represent combinatorially?

Investigate $F(x) = 1/(1-x)$.

$$F(G(x)) = \frac{1}{1-G(x)} = 1 + G(x) + G(x)^2 + G(x)^3 + \cdots .$$

- This is an infinite sum of (likely infinite) power series.
- The constant term h_0 of $H(x)$ only makes sense if .
- This implies that x^n divides $G(x)^n$.
  Hence, there are at most $n-1$ summands which contain x^{n-1}.
  We conclude that the infinite sum makes sense.

For a general composition with $g_0 = 0$,

$$F(G(x)) = \sum_{n\geq 0} f_n G(x)^n = f_0 + f_1 G(x) + f_2 G(x)^2 + f_3 G(x)^3 + \cdots .$$

Compositions. of. Generating Functions.

Interpreting $\dfrac{1}{1-G(x)} = 1 + G(x) + G(x)^2 + G(x)^3 + \cdots$:

Recall: If $G(x)$ represents the number of ways to build a certain structure on an n-element set, then $G(x)^k$ represents the number of ways to split the n-element set into an list of k pieces and build a structure on each piece.

Conclusion: As long as $g_0 = 0$, then $1 + G(x) + G(x)^2 + G(x)^3 + \cdots$ represents all ways to break down an n-element set into any number of (ordered) pieces and build a structure on each piece.

A **partition**: $p_1 + p_2 + \cdots + p_k = n$ with $p_1 \geq p_2 \geq \cdots \geq p_k$.

A **composition**: $c_1 + c_2 + \cdots + c_k = n$ with no restrictions.

Think: A composition of generating functions equals
a composition. of. generating. functions.

An Example, Compositions

Example. How many compositions of n are there?

Solution. We break down "n" into pieces of size k.

What is the "structure" on each piece?

For a piece of size $k > 0$, there is one option—it's a piece of size k.

Therefore $G(x) = x + x^2 + x^3 + x^4 + \cdots =$

The generating function for compositions is
$H(x) = \frac{1}{1-G(x)} =$

So the number of compositions of n is

A Composition Example

Example. How many ways are there to take a group of n soldiers, break them into non-empty platoons and assign each platoon a leader?

Solution. We break down the n soldiers set into groups of size k.

What is the "structure" on each group?

For a group of size $k > 0$, there are ______ options.

Therefore $G(x) =$

And the generating function for such a military breakdown is

$$H(x) = \frac{1}{1 - G(x)} = \frac{1 - 2x + x^2}{1 - 3x + x^2}$$