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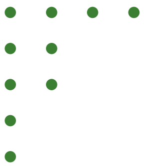
Example. The Ferrers diagram of $42211 \vdash 10$ is



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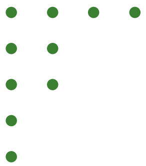

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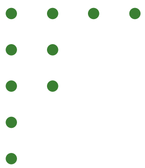

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Some partitions are **self-conjugate**, satisfying $\lambda = \lambda^c$.

A generating function for partitions

Recall from our basketball example: The generating function for the number of ways to partition an integer into parts of size 1, 2, or 3 is

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A formula for integer partitions

Bruinier and Ono (2011) found an algebraic formula for the partition function $P(n)$ as a finite sum of algebraic numbers as follows. Define the weight-2 meromorphic modular form $F(z)$ by

$$F(z) = \frac{1}{2} \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta^2(z)\,\eta^2(2z)\,\eta^2(3z)\,\eta^3(6z)}, \qquad (27)$$

were $q = e^{2\pi i z}$, $E_2(q)$ is an Eisenstein series, and $\eta(q)$ is a Dedekind eta function. Now define

$$R(z) = -\left(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi y}\right) F(z), \qquad (28)$$

where $z = x + iy$. Additionally let Q_n be any set of representatives of the equivalence classes of the integral binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ such that $6 \mid a$ with $a > 0$ and $b \equiv 1 \pmod{12}$, and for each $Q(x, y)$, let α_Q be the so-called CM point in the upper half-plane, for which $Q(\alpha_Q, 1) = 0$. Then

$$P(n) = \frac{\text{Tr}(n)}{24n - 1}, \qquad (29)$$

where the trace is defined as

$$\text{Tr}(n) = \sum_{Q \in Q_n} R(\alpha_Q). \qquad (30)$$

Weisstein, Eric W. "Partition Function P."
From MathWorld—A Wolfram Web Resource.
http://mathworld.wolfram.com/PartitionFunctionP.html

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- There is a beautiful generating function, but **no nice formula**!
- Finding a generating function for a subset of partitions is easy if you understand each factor in the product.

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Example. **THE FOLLOWING AMAZING FACT!!!!1!!!11!!**

The number of partitions of n using only odd parts, o_n	=	The number of partitions of n using distinct parts, d_n

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See, I told you they were equal. □

A recurrence relation for $P(n, k)$ (p.78)

Example. Prove a recurrence relation for $P(n, k)$:

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The same bijection gives:

Theorem 4.4.2. ____________________ equals $P(n, \textit{largest part} \leq k)$.

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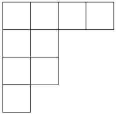
Simplifying, $\frac{1}{6}\binom{n-1}{2} + \frac{1}{4}\binom{n-2}{1} = \frac{n^2}{12} - \frac{1}{3} + \cdots$

Standard Young Tableaux

Related to some current lines of research in algebra and combinatorics:

A **Young diagram** is a representation of a partition using left-justified boxes.



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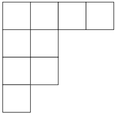

A **standard Young tableau** is a placement of the integers 1 through n into the boxes, where the numbers in both the rows and the columns are increasing.

1	3	5	9
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7			

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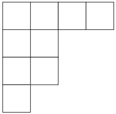

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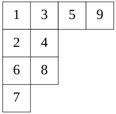
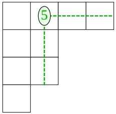


The **hook length** $h(i, j)$ of a cell (i, j) is the number of cells in the "hook" to the left and down.



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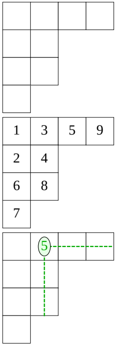

Question: How many SYT are there of shape $\lambda \vdash n$?

Answer: $$\frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$$