More about partitions

- $3+1+1$, $1+3+1$, and $1+1+3$ are all the same partition, so we will write the numbers in non-increasing order.
- We use greek letters to denote partitions, often λ ("lambda"), μ ("mu"), and ν ("nu").
- We'll write: $\lambda : n = n_1 + n_2 + \cdots + n_k$ or $\lambda \vdash n$.

For example, $\lambda : 5 = 3+1+1$, or $\lambda = 311$, or $\lambda = 3^1 1^2$, or $311 \vdash 5$.

A pictoral representation of $\lambda = n_1 n_2 \cdots n_k$ is its *Ferrers diagram*, a left-justified array of dots with k rows, containing n_i dots in row i.

Example. The Ferrers diagram of $42211 \vdash 10$ is

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & \bullet & & \\ \bullet & & & \\ \bullet & & & \end{array}$$

The **conjugate** of a partition λ is the partition λ^c which interchanges rows and columns.

Some partitions are **self-conjugate**, satisfying $\lambda = \lambda^c$.

A generating function for partitions

Recall from our basketball example: The generating function for the number of ways to partition an integer into parts of size 1, 2, or 3 is

$$\frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}$$

If we include parts of any size, we infer:

Let $p(n)$ be the number of partitions of the integer n. Then

$$\sum_{n\geq 0} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Notes:

- Infinite product! But, for any n only finitely many terms involved.
- There is a beautiful generating function, but **no nice formula**!
- Finding a generating function for a subset of partitions is easy if you understand each factor in the product.

A formula for integer partitions

Bruinier and Ono (2011) found an algebraic formula for the partition function $P(n)$ as a finite sum of algebraic numbers as follows. Define the weight-2 meromorphic modular form $F(z)$ by

$$F(z) = \frac{1}{2} \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta^2(z)\eta^2(2z)\eta^2(3z)\eta^3(6z)}, \tag{27}$$

were $q = e^{2\pi i z}$, $E_2(q)$ is an Eisenstein series, and $\eta(q)$ is a Dedekind eta function. Now define

$$R(z) = -\left(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi y}\right)F(z), \tag{28}$$

where $z = x + iy$. Additionally let Q_n be any set of representatives of the equivalence classes of the integral binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ such that $6 \mid a$ with $a > 0$ and $b \equiv 1 \pmod{12}$, and for each $Q(x, y)$, let α_Q be the so-called CM point in the upper half-plane, for which $Q(\alpha_Q, 1) = 0$. Then

$$P(n) = \frac{\mathrm{Tr}(n)}{24n - 1}, \tag{29}$$

where the trace is defined as

$$\mathrm{Tr}(n) = \sum_{Q \in Q_n} R(\alpha_Q). \tag{30}$$

Weisstein, Eric W. "Partition Function P."
From MathWorld—A Wolfram Web Resource.
`http://mathworld.wolfram.com/PartitionFunctionP.html`

Partitions: odd parts and distinct parts

Example. <u>**THE FOLLOWING AMAZING FACT!!!!1!!!11!!**</u>

The number of partitions of n using only odd parts, o_n	$=$	The number of partitions of n using distinct parts, d_n

Investigation: Does this make sense? For $n = 6$,

o_6: d_6:

Solution. Determine the generating functions

$O(x) = \sum_{n \geq 0} o_n x^n$ $D(x) = \sum_{n \geq 0} d_n x^n$

See, I told you they were equal. $\square$

A recurrence relation for $P(n, k)$ (p.78)

Example. Prove a recurrence relation for $P(n, k)$:

$$P(n, k) = P(n-1, k-1) + P(n-k, k)$$

Question: How many partitions of n are there into k parts?

LHS: $P(n, k)$

RHS: Condition on whether the smallest part is of size 1.

- If so, there are $P(n-1, k-1)$ partitions via the bijection

$$f : \left\{ \begin{array}{c} \text{partitions of } n \text{ into } k \text{ parts} \\ \text{with smallest part } 1. \end{array} \right\} \to \left\{ \begin{array}{c} \text{partitions of } n-1 \\ \text{into } k-1 \text{ parts.} \end{array} \right\}.$$

- If not: there are $P(n-k, k)$ partitions via the bijection

$$g : \left\{ \begin{array}{c} \text{partitions of } n \text{ into } k \text{ parts} \\ \text{with smallest part } \neq 1. \end{array} \right\} \to \left\{ \begin{array}{c} \text{partitions of } n-k \\ \text{into } k \text{ parts.} \end{array} \right\}.$$

Using conjugation

Theorem 4.4.1. $P(n, k)$ equals $P(n, \textit{largest part} = k)$

Proof. The conjugation function $f : \lambda \to \lambda^c$ is a bijection

$$f : \left\{ \begin{array}{c} \text{partitions of } n \\ \text{into exactly } k \text{ parts} \end{array} \right\} \to \left\{ \begin{array}{c} \text{partitions of } n \text{ with} \\ \text{largest part of size } k. \end{array} \right\}.$$

The same bijection gives:

Theorem 4.4.2. ____________________ equals $P(n, \textit{largest part} \leq k)$.

Characterization of self-conjugate partitions

Theorem 4.4.3. $P(n, \textit{self conjugate}) = P(n, \textit{distinct odd parts})$

Proof. Define a bijection which "unfolds" self-conjugate partitions:

$$f : \left\{ \begin{array}{c} \text{self-conjugate} \\ \text{partitions } \lambda \text{ of } n \end{array} \right\} \to \left\{ \begin{array}{c} \text{partitions } \mu \text{ of } n \text{ into} \\ \text{distinct odd parts} \end{array} \right\}.$$

- Define parts of μ by **unpeeling** λ layer by layer.
- Iteratively remove the first row and first column of λ.

Question: Is f well defined?

Define the inverse function $g = f^{-1} : \mu \mapsto \lambda$:

- Find the **center dot** of each part μ_i.
- **Fold** each μ_i about its center dot.
- **Nest** these folded parts to create λ.

Question: Is g well defined?

Question: Is $g(f(\lambda)) = \lambda$?

Partition Formulas

A formula for $P(n)$ is hard. We can find formulas for $P(n,2)$ & $P(n,3)$.

Theorem. $P(n,2) =$ ______.

Proof.

Theorem 4.4.5. $P(n,3)$ is the closest integer to $n^2/12$.

Proof. Above theorems give us:
$P(n,3) = P(n-3, \text{at most 3 parts}) = P(n-3, \text{parts of size} \leq 3)$.

Question: How many partitions of $n-3$ into parts of size 1, 2, 3?

Answer: The generating function is

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{1/6}{(1-x)^3} + \frac{1/4}{(1-x)^2} + \frac{1/3}{1-x^3} + \frac{1/4}{1-x^2}.$$

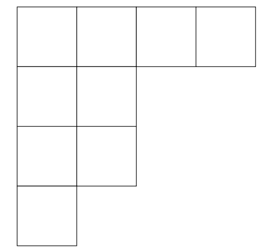
The coefficient of x^{n-3} is $\frac{1}{6}\left(\!\binom{3}{n-3}\!\right) + \frac{1}{4}\left(\!\binom{2}{n-3}\!\right) + [\frac{1}{3} \text{ or } 0] + [\frac{1}{4} \text{ or } 0]$.

Simplifying, $\frac{1}{6}\binom{n-1}{2} + \frac{1}{4}\binom{n-2}{1} = \frac{n^2}{12} - \frac{1}{3} + \cdots$

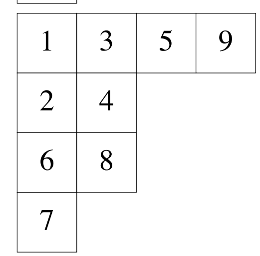
Standard Young Tableaux

Related to some current lines of research in algebra and combinatorics:

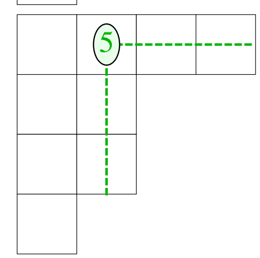
A **Young diagram** is a representation of a partition using left-justified boxes.



A **standard Young tableau** is a placement of the integers 1 through n into the boxes, where the numbers in both the rows and the columns are increasing.



The **hook length** $h(i, j)$ of a cell (i, j) is the number of cells in the "hook" to the left and down.



Question: How many SYT are there of shape $\lambda \vdash n$?

Answer: $$\frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$$