More about partitions

- $3 + 1 + 1$, $1 + 3 + 1$, and $1 + 1 + 3$ are all the same partition, so we will write the numbers in non-increasing order.

- We use greek letters to denote partitions, often $\lambda$ ("lambda"), $\mu$ ("mu"), and $\nu$ ("nu").

- We’ll write: $\lambda : n = n_1 + n_2 + \cdots + n_k$ or $\lambda \vdash n$.

For example, $\lambda : 5 = 3 + 1 + 1$, or $\lambda = 311$, or $\lambda = 3^11^2$, or $311 \vdash 5$.

A pictorial representation of $\lambda = n_1n_2\cdots n_k$ is its Ferrers diagram, a left-justified array of dots with $k$ rows, containing $n_i$ dots in row $i$.

Example. The Ferrers diagram of $42211 \vdash 10$ is

The conjugate of a partition $\lambda$ is the partition $\lambda^c$ which interchanges rows and columns.

Some partitions are self-conjugate, satisfying $\lambda = \lambda^c$. 
A generating function for partitions

Recall from our basketball example: The generating function for the number of ways to partition an integer into parts of size 1, 2, or 3 is

$$\frac{1}{(1-x)} \frac{1}{(1-x^2)} \frac{1}{(1-x^3)}$$

If we include parts of any size, we infer:
Let $p(n)$ be the number of partitions of the integer $n$. Then

$$\sum_{n\geq0} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Notes:

- Infinite product! But, for any $n$ only finitely many terms involved.
- There is a beautiful generating function, but no nice formula!
- Finding a generating function for a subset of partitions is easy if you understand each factor in the product.
A formula for integer partitions

Bruinier and Ono (2011) found an algebraic formula for the partition function $P(n)$ as a finite sum of algebraic numbers as follows. Define the weight-2 meromorphic modular form $F(z)$ by

$$ F(z) = \frac{1}{2} \frac{E_2(z) - 2 E_2(2z) - 3 E_2(3z) + 6 E_2(6z)}{\eta^2(z) \eta^2(2z) \eta^2(3z) \eta^3(6z)}, $$

(27)

were $q = e^{2\pi i z}$, $E_2(q)$ is an Eisenstein series, and $\eta(q)$ is a Dedekind eta function. Now define

$$ R(z) = -\left( \frac{1}{2\pi i} \frac{d}{dz} + \frac{1}{2\pi y} \right) F(z), $$

(28)

where $z = x + iy$. Additionally let $\mathcal{Q}_n$ be any set of representatives of the equivalence classes of the integral binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ such that $6 | a$ with $a > 0$ and $b \equiv 1 \pmod{12}$, and for each $Q(x, y)$, let $\alpha_Q$ be the so-called CM point in the upper half-plane, for which $Q(\alpha_Q, 1) = 0$. Then

$$ P(n) = \frac{\text{Tr}(n)}{24n - 1}, $$

(29)

where the trace is defined as

$$ \text{Tr}(n) = \sum_{Q \in \mathcal{Q}_n} R(\alpha_Q). $$

(30)

Example. **THE FOLLOWING AMAZING FACT!!!!!1!!!!11!!**

<table>
<thead>
<tr>
<th>The number of partitions of $n$ using only odd parts, $o_n$</th>
<th>=</th>
<th>The number of partitions of $n$ using distinct parts, $d_n$</th>
</tr>
</thead>
</table>

Investigation: Does this make sense? For $n = 6$,

$\begin{align*}
o_6: \\
d_6: 
\end{align*}$

**Solution.** Determine the generating functions

\[
O(x) = \sum_{n \geq 0} o_n x^n \quad \quad D(x) = \sum_{n \geq 0} d_n x^n
\]

See, I told you they were equal. □
Example. Prove a recurrence relation for $P(n, k)$:

$$P(n, k) = P(n - 1, k - 1) + P(n - k, k)$$

**Question:** How many partitions of $n$ are there into $k$ parts?

**LHS:** $P(n, k)$

**RHS:** Condition on whether the smallest part is of size 1.

- **If so:** there are $P(n - 1, k - 1)$ partitions via the bijection

  $$f : \begin{cases} 
  \text{partitions of } n \text{ into } k \text{ parts} \\
  \quad \text{with smallest part 1.} 
  \end{cases} \rightarrow \begin{cases} 
  \text{partitions of } n - 1 \\
  \quad \text{into } k - 1 \text{ parts.} 
  \end{cases}.$$

- **If not:** there are $P(n - k, k)$ partitions via the bijection

  $$g : \begin{cases} 
  \text{partitions of } n \text{ into } k \text{ parts} \\
  \quad \text{with smallest part } \neq 1. 
  \end{cases} \rightarrow \begin{cases} 
  \text{partitions of } n - k \\
  \quad \text{into } k \text{ parts.} 
  \end{cases}.$$
Using conjugation

**Theorem 4.4.1.** \( P(n, k) \) equals \( P(n, \text{largest part } = k) \)

**Proof.** The conjugation function \( f : \lambda \rightarrow \lambda^c \) is a bijection

\[
f : \left\{ \begin{array}{c}
\text{partitions of } n \\
\text{into exactly } k \text{ parts}
\end{array} \right\} \rightarrow \left\{ \begin{array}{c}
\text{partitions of } n \text{ with } \\
\text{largest part of size } k.
\end{array} \right\}.
\]

The same bijection gives:

**Theorem 4.4.2.** \( \text{_______________} \) equals \( P(n, \text{largest part } \leq k) \).
Characterization of self-conjugate partitions

**Theorem 4.4.3.** $P(n, \text{self conjugate}) = P(n, \text{distinct odd parts})$

**Proof.** Define a bijection which “unfolds” self-conjugate partitions:

$$f : \left\{ \begin{array}{l} \text{self-conjugate} \\
\text{partitions } \lambda \text{ of } n \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{partitions } \mu \text{ of } n \text{ into} \\
\text{distinct odd parts} \end{array} \right\}. $$

- Define parts of $\mu$ by **unpeeling** $\lambda$ layer by layer.
- Iteratively remove the first row and first column of $\lambda$.

**Question:** Is $f$ well defined?

Define the inverse function $g = f^{-1} : \mu \mapsto \lambda$:

- Find the **center dot** of each part $\mu_i$.
- **Fold** each $\mu_i$ about its center dot.
- **Nest** these folded parts to create $\lambda$.

**Question:** Is $g$ well defined?

**Question:** Is $g(f(\lambda)) = \lambda$?
Partition Formulas

A formula for $P(n)$ is hard. We can find formulas for $P(n, 2)$ & $P(n, 3)$.

**Theorem.** $P(n, 2) = \underline{\hspace{2cm}}$.

**Proof.**

**Theorem 4.4.5.** $P(n, 3)$ is the closest integer to $n^2/12$.

**Proof.** Above theorems give us:

$P(n, 3) = P(n - 3, \text{at most 3 parts}) = P(n - 3, \text{parts of size} \leq 3)$.

**Question:** How many partitions of $n - 3$ into parts of size 1, 2, 3?

**Answer:** The generating function is

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{1}{6} \left( \frac{3}{(n-3)} \right) + \frac{1}{4} \left( \frac{2}{n-3} \right) + \left[ \frac{1}{3} \text{ or 0} \right] + \left[ \frac{1}{4} \text{ or 0} \right].$$

The coefficient of $x^{n-3}$ is

$$\frac{1}{6} \binom{n-1}{2} + \frac{1}{4} \binom{n-2}{1} = \frac{n^2}{12} - \frac{1}{3} + \cdots.$$
Standard Young Tableaux

Related to some current lines of research in algebra and combinatorics:

A **Young diagram** is a representation of a partition using left-justified boxes.

A **standard Young tableau** is a placement of the integers 1 through $n$ into the boxes, where the numbers in both the rows and the columns are increasing.

The **hook length** $h(i, j)$ of a cell $(i, j)$ is the number of cells in the “hook” to the left and down.

**Question:** How many SYT are there of shape $\lambda \vdash n$?

**Answer:**

$$\frac{n!}{\prod_{(i,j) \in \lambda} h(i, j)}$$