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Solution. Use generating functions: define $A(x) = \sum_{k\geq 0} a_k x^k$.

Step 1: Multiply both sides of the recurrence by x^{k+1} and sum over all k:

$$\sum_{k\geq 0} a_{k+1}x^{k+1} = \sum_{k\geq 0}(2a_k + 1)x^{k+1}$$

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Step 2: Massage the sums to find copies of $A(x)$.
LHS: Re-index, find missing term; RHS: separate into pieces.

$$\sum_{k\geq 1} a_k x^k = \sum_{k\geq 0} 2a_k x^{k+1} + \sum_{k\geq 0} x^{k+1}$$

Conversion to functions of $A(x)$:

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Step 3: Solve for the compact form of $A(x)$.

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Step 4: Extract the coefficients.

When the degree of the numerator is smaller than the degree of the denominator, we can use partial fractions to determine an expression for $A(x)$ of the form:

$$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-x}$$

Solving gives $A(x) = \frac{1}{1-2x} + \frac{-1}{1-x}$; each of which can be expanded:

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$$A(x) = \sum_{k\geq 0} 2^k x^k + \sum_{k\geq 0} (-1)\, x^k = \sum_{k\geq 0} (2^k - 1)\, x^k$$

Therefore, $a_k = 2^k - 1$.

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Therefore, $F(x) - x - 0 = x\big(F(x) - 0\big) + x^2 F(x)$, so

$$F(x) = \frac{x}{1 - x - x^2}$$

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So the Fibonacci numbers have generating function $x/(1-x-x^2)$.
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Practicality: $(1+\sqrt{5})/2 \approx 1.61803$ and 1 mi ≈ 1.609344 km

Solving recurrence relations with repeated roots

With repeated roots in the denominator, the result is not quite as nice.

Example. Find the partial fraction decomposition of $\frac{x}{(1-2x)^2(1+5x)}$.

Since $(1-2x)^2$ is a repeated root,

$$\frac{x}{(1-2x)^2(1+5x)} = \frac{A}{(1-2x)} + \frac{B}{(1-2x)^2} + \frac{C}{(1+5x)}.$$

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Clearing the denominator gives:

$$x = A(1-2x)(1+5x) + B(1+5x) + C(1-2x)^2.$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 0 + B(1+\frac{5}{2}) + 0$; so $B = \frac{1}{7}$.
When $x = -\frac{1}{5}$, $-\frac{1}{5} = 0 + 0 + C(1+\frac{2}{5})^2$; so $C = \frac{-5}{49}$.

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$$\frac{x}{(1-2x)^2(1+5x)} = \frac{-\frac{2}{49}}{(1-2x)} + \frac{\frac{7}{49}}{(1-2x)^2} + \frac{-\frac{5}{49}}{(1+5x)}.$$

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Example. Let $\{h_n\}_{n\geq 0}$ be a sequence satisfying

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0,$$

with initial conditions $h_0 = 1$, $h_1 = 1$, and $h_2 = -1$.

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Find the generating function and formula for h_n.

$$\begin{array}{rcl} h(x) & = & h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots + h_nx^n + \cdots , \\ +xh(x) & = & h_0x + h_1x^2 + h_2x^3 + \cdots + h_{n-1}x^n + \cdots , \\ -16x^2h(x) & = & -16h_0x^2 - 16h_1x^3 + \cdots - 16h_{n-2}x^n + \cdots , \\ +20x^3h(x) & = & 20h_0x^3 + \cdots + 20h_{n-3}x^n + \cdots , \\ \hline \end{array}$$

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Therefore $\frac{1}{(1-2x)^2} = \sum_{n\geq 0} \binom{n+1}{n} (2x)^n$

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