Generating functions

"A generating function is a clothesline on which we hang up a sequence of numbers for display."

— *Generatingfunctionology*, H. S. Wilf

Definition: For any sequence $\{a_n\}_{n\geq 0} = a_0, a_1, a_2, a_3, \ldots$, its generating function is the formal power series

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{k\geq 0} a_k x^k.$$

Example. Let f_n be the Fibonacci numbers. Then

$$F(x) = \sum_{k\geq 0} f_k x^k = 0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots .$$

We will see that we can simplify this expression greatly. In fact,

$$F(x) = x/\left(1 - x - x^2\right).$$

I will call this the compact form of the generating function.

Why Generating Functions?

We will use generating functions to:

- Find an exact formula for the terms of a sequence.
- Prove identities involving sequences.
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Others use generating functions to:

- Use complex analysis to solve combinatorial problems.
- Understand the asymptotics of a sequence.
- Find averages and statistical properties.
- Understand **something** about a sequence.

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$$3+3 \qquad 3+2+1 \qquad 3+1+1+1 \qquad 2+2+2$$

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We'll first reanalyze the question of scoring six points and then generalize to larger numbers.

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To start, break down the possible ways of getting six points total into one-point, two-point, and three-point shots.

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How many points could be scored using one-point shots?

0 pts	or	1 pt	or	2 pts	or	3 pts	or	4 pts	or	5 pts	or	6 pts
x^0	$+$	x^1	$+$	x^2	$+$	x^3	$+$	x^4	$+$	x^5	$+$	x^6

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How many points could be scored using two-point shots?

How many points could be scored using three-point shots?

Multiply these algebraic expressions together:

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 7x^7 + 8x^8 + 8x^9 + 8x^{10} + 7x^{11} + 7x^{12} + 5x^{13} + 4x^{14} + 3x^{15} + 2x^{16} + x^{17} + x^{18}$$

and find the coefficient of the x^6 term.

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Why does this work? A score a from 1-pt, b from 2-pt, c from 3-pt, gives a term in the product of $x^a x^b x^c = x^{a+b+c}$. Collecting like terms makes the coefficient of x^k the number of ways to score k points. (x^{15}?)

Generating function example: Basketball

In order to take into account all the ways to score 98 points, we include more terms in each factor:

One-point shots: $1 + x + x^2 + \cdots + \qquad = \underline{\hspace{3cm}}.$

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Conclusion: The generating function for the number of ways to score any number of points in basketball is

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Notation: $[x^k]f(x)$ is the coefficient of x^k in the expansion of the generating function $f(x)$. Example. $[x^{98}]b(x) = 850$.

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$$(1+x)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k \qquad e^x = \sum_{k\geq 0} \frac{1}{k!} x^k$$

Manipulations on $A(x) = \sum_{k \geq 0} a_k x^k$

Question: How can we simplify $[x^k]\big(x^b A(x)\big)$?

Example. $[x^{10}] \left(\frac{x^5}{(1-2x)} + \frac{1}{1-x} \right)$

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Example. What is the compact form of $\sum_{k \geq 0} (-3)^{k+2} x^k$?

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- The number of bananas is a multiple of five.
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Strategy. Write down a power series for each piece of fruit, multiply them together, and extract the coefficient of x^n.

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Die F: $\{1,2,2,3,3,4\}$ and die G: $\{1,3,4,5,6,8\}$

Memories of calculus. . .

With formal power series, we interchange derivatives, integrals, sums.

$$\sum_{k\geq 0} \frac{d}{dx} x^k = \frac{d}{dx} \sum_{k\geq 0} x^k \qquad .$$

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Example. Find $\displaystyle\sum_{n\geq 0} \frac{n^2+4n+5}{n!}$