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Definition: $P(k, i)$ is the number of partitions of k into i parts.

Example. We saw $P(6, 1) = 1$, $P(6, 2) = 3$, $P(6, 3) = 3$, $P(6, 4) = 2$, $P(6, 5) = 1$, and $P(6, 6) = 1$.

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Definition: $P(k)$ is the number of partitions of k into any number of parts.

Example. $P(6) = 1 + 3 + 3 + 2 + 1 + 1 = 11$.

THE CHART, COMPLETED

Question: In how many ways can we place k objects in n boxes?

Distributions of		Restrictions on # objects received			
k **objects**	n **boxes**	none	≤ 1	≥ 1	$= 1$
distinct	distinct	n^k	$(n)_k$	$n!S(k,n)$	$n!$ or 0
identical	distinct	$\left(\!\binom{n}{k}\!\right)$	$\binom{n}{k}$	$\left(\!\binom{n}{k-n}\!\right)$	1 or 0
distinct	identical	$\sum S(k,i)$	1 or 0	$S(k,n)$	1 or 0
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Principle of Inclusion-Exclusion

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Solution.

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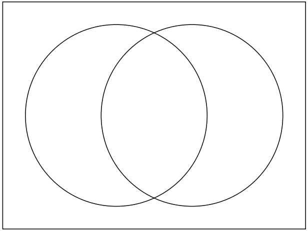
Example. Suppose that in this class, 14 students play soccer and 17 students play basketball. How many students play a sport?

Solution.

Let S be the set of students who play soccer
and B be the set of students who play basketball.
Then, $|S \cup B| = |S| + |B|$ __________.



Principle of Inclusion-Exclusion

When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ ($\mathcal{U}$ for universe) and the sets A_i are *pairwise disjoint*, we have $|A| = |A_1| + \cdots + |A_k|$.

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&\quad - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|
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|A_1 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum \left|A_i \cap A_j \cap A_k\right| \cdots
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It may be more convenient to apply inclusion/exclusion where the A_i are *forbidden* subsets of $\mathcal{U}$, in which case ________________.

mmm. . .PIE

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And finally: So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

Combinations with Repetitions

Quick review

1. How many ways are there to choose k elements out of the set $\{1 \cdot a_1, 1 \cdot a_2, \cdots, 1 \cdot a_n\}$?

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What we would like to calculate is:

> In how many ways can we choose k elements out of an arbitrary multiset?

Now, it's as easy as PIE.

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Definition: An n-derangement is an n-permutation $\pi = p_1 p_2 \cdots p_n$ such that $p_1 \neq 1$, $p_2 \neq 2$, $\cdots$, $p_n \neq n$.

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Notation: We let D_n be the number of all n-derangements.

When you see D_n, think combinatorially: "The number of ways to return n hats to n people so no one gets his/her own hat back"

Calculating the number of derangements

Example. Calculate D_n.

Solution. Let $\mathcal{U}$ be the set of all n-permutations.
Remove bad permutations using PIE.
For all i from 1 to n, define A_i to be n-perms where $p_i = i$.

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When intersecting k sets, $|A_{i_1} \cap \cdots \cap A_{i_k}| =$

Recall: $|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum \left|A_i \cap A_j \cap A_k\right| \cdots$

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Therefore, $D_n = |\mathcal{U}| - |A_1 \cup \cdots \cup A_n| =$

Randomly returning hats

Upon simplification, we see

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! \\ &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right] \end{aligned}$$

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$$D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0!$$
$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!}$$
$$= n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

Recall: Taylor series expansion of e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

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Recall: Taylor series expansion of e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Plug in $x = -1$ and truncate after n terms to see that

$$e^{-1} \approx \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right]$$

Randomly returning hats

Upon simplification, we see

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n\binom{n}{n}0! \\ &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n\frac{n!}{n!} \\ &= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right] \end{aligned}$$

Recall: Taylor series expansion of e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Plug in $x = -1$ and truncate after n terms to see that

$$e^{-1} \approx \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right]$$

Conclusion: If n people go to a party and the hats are passed back randomly, the probability that no one gets his or her hat back at the party is $D_n/n!$, which is approximately $1/e \approx 37\%$.

Combinatorial proof involving D_n

Recall: The combinatorial interpretation of D_n is: "The number of ways to return n hats to n people so no one gets his/her own hat back"

Example. Prove the following recurrence relation for D_n combinatorially.

$$D_n = (n-1)\big(D_{n-2} + D_{n-1}\big)$$

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We calculate: $|\mathcal{U}| = k^n$, $|A_i| = (k-1)^n$, $|A_i \cap A_j| = (k-2)^n$
When intersecting j sets, $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k-j)^n$.

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Therefore, $k!S(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$;
we conclude $S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$.