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Stirling numbers

The **Stirling number of the second kind** counts the number of ways to partition a set of k elements into i non-empty subsets.

Notation: $S(k, i)$ or $\left\{ k \atop i \right\}$. ← **Careful about this order!**

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0	1							
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In Stirling's triangle:

$S(k,1) = S(k,k) = 1.$

$S(k,2) = 2^{k-1} - 1.$

$S(k,k-1) = \binom{k}{2}.$

Later: Formula for $S(k,i)$.

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Ask: In how many ways can we place k objects into i boxes?

We'll condition on the placement of element $\#i$:

THE CHART

Question: In how many ways can we place k objects in n boxes?

Distributions of		Restrictions on # objects received			
k **objects**	n **boxes**	none	≤ 1	≥ 1	$= 1$
distinct	distinct	n^k	$(n)_k$		$n!$ or 0
identical	distinct	$\left(\!\binom{n}{k}\!\right)$	$\binom{n}{k}$	$\left(\!\binom{n}{k-n}\!\right)$	1 or 0
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$S(k, n)$ counts ways to place k distinct obj. into n identical boxes.

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Bell numbers

Definition: The **Bell number** B_k is the number of partitions of a set with k elements, into any number of non-empty parts.

We have $B_k = S(k,0) + S(k,1) + S(k,2) + \cdots + S(k,k)$.

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Theorem 2.3.3. The Bell numbers satisfy a recurrence:

$$B_k = \binom{k-1}{0} B_0 + \binom{k-1}{1} B_1 + \cdots + \binom{k-1}{k-1} B_{k-1}.$$

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Proof: How many partitions of $\{1, \ldots, k\}$ are there?

LHS: B_k, obviously.

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RHS: Condition on the box containing the last element k:
How many partitions of $[k]$ contain i elements in the box with k?