Distinct objects in indistinguishable boxes

When placing k distinguishable objects into n indistinguishable boxes, what matters? ________________________________

- Each object needs to be in some box.
- No object is in two boxes.

We have rediscovered ____________________.

So ask "How many set partitions are there of a set with k objects?"

Or even, "How many set partitions are there of k objects into n parts?"

In the homework, you will see ...

The answer is not straightforward.

Stirling numbers

The **Stirling number of the second kind** counts the number of ways to partition a set of k elements into i non-empty subsets.

Notation: $S(k,i)$ or $\left\{ {k \atop i} \right\}$. ← **Careful about this order!**

k	$\left\{ {k \atop 0} \right\}$	$\left\{ {k \atop 1} \right\}$	$\left\{ {k \atop 2} \right\}$	$\left\{ {k \atop 3} \right\}$	$\left\{ {k \atop 4} \right\}$	$\left\{ {k \atop 5} \right\}$	$\left\{ {k \atop 6} \right\}$	$\left\{ {k \atop 7} \right\}$
0	1							
1		1						
2		1	1					
3		1	3	1				
4		1	7	6	1			
5		1	15	25	10	1		
6		1	31	90	65	15	1	
7		1						1

In Stirling's triangle:

$S(k,1) = S(k,k) = 1.$

$S(k,2) = 2^{k-1} - 1.$

$S(k,k-1) = \binom{k}{2}.$

Later: Formula for $S(k,i)$.

To fill in the table, find a recurrence for $S(k,i)$:

Ask: In how many ways can we place k objects into i boxes?
We'll condition on the placement of element $\#i$:

THE CHART

Question: In how many ways can we place k objects in n boxes?

Distributions of		Restrictions on # objects received			
k **objects**	n **boxes**	none	≤ 1	≥ 1	$= 1$
distinct	distinct	n^k	$(n)_k$		$n!$ or 0
identical	distinct	$\left(\!\binom{n}{k}\!\right)$	$\binom{n}{k}$	$\left(\!\binom{n}{k-n}\!\right)$	1 or 0
distinct	identical				
identical	identical				

$S(k, n)$ counts ways to place k distinct obj. into n identical boxes.

What if we then label the boxes?

How many ways to distribute distinct objects into identical boxes

- If there is exactly one item in each box?
- If there is at most one item in each box?
- What about with no restrictions?

Bell numbers

Definition: The **Bell number** B_k is the number of partitions of a set with k elements, into any number of non-empty parts.

We have $B_k = S(k,0) + S(k,1) + S(k,2) + \cdots + S(k,k)$.

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
1	1	2	5	15	52	203	877	4140	21147

Theorem 2.3.3. The Bell numbers satisfy a recurrence:

$$B_k = \binom{k-1}{0} B_0 + \binom{k-1}{1} B_1 + \cdots + \binom{k-1}{k-1} B_{k-1}.$$

Proof: How many partitions of $\{1, \ldots, k\}$ are there?

LHS: B_k, obviously.

RHS: Condition on the box containing the last element k:
How many partitions of $[k]$ contain i elements in the box with k?