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- What do the objects look like?
  - Do the objects all look the same?
- What do the boxes look like?
  - Do the boxes all look the same?
- Are there any restrictions?
  - Is there a size limit?
  - Must there be an object in each box?

# Counting distributions

*Definition:* A **distribution** is an assignment of objects to recipients.

Certain counting problems can be revisited in this framework:

$$\left\{\begin{array}{c}\text{Five-letter passwords}\\ \text{on } \{A, B, C, D, E, F, G\}\end{array}\right\} \text{ correspond to } \left\{\begin{array}{c}\text{Distributions of}\\ \text{\_\_\_ distinct objects}\\ \text{into \_\_\_ distinct boxes}\end{array}\right\}$$

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|---|---|---|---|---|---|
| $k$ **objects** | $n$ **boxes** | none | $\leq 1$ | $\geq 1$ | $= 1$ |
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| identical | distinct | $\left(\!\binom{n}{k}\!\right)$ | $\binom{n}{k}$ | $\left(\!\binom{n}{k-n}\!\right)$ | 1 or 0 |
| distinct | identical | | | | |
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# Pascal's triangle

Pascal's identity gives us the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
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| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 1 | | | | | | |
| 2 | 1 | | 1 | | | | | |
| 3 | 1 | | | 1 | | | | |
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Seq's in Pascal's triangle:

- $1, 2, 3, 4, 5, \ldots$ $\binom{n}{1}$ ($a_n = n$)
- $1, 3, 6, 10, 15, \ldots$ $\binom{n}{2}$ triangular
- $1, 4, 10, 20, 35, \ldots$ $\binom{n}{3}$ tetrahedral
- $1, 2, 6, 20, 70, \ldots$ $\binom{2n}{n}$ centr. binom.

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Online Encyclopedia of Integer Sequences:

`http://www.research.att.com/~njas/sequences/`

# Binomial Theorem

**Theorem 2.2.2.** Let $n$ be a positive integer. For all $x$ and $y$,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + y^n.$$

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Rewrite in summation notation!
Determine the generic term $[\binom{n}{k}x \quad y \quad]$ and the bounds on $k$

$$(x+y)^n = \sum$$

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Proof. In the expansion of $(x+y)(x+y)\cdots(x+y)$, in how many ways can a term have the form $x^{n-k}y^k$?

From the $n$ factors $(x+y)$, you must choose a "$y$" exactly $k$ times. Therefore, $\binom{n}{k}$ ways. We recover the desired equation. □

# Counting integral solutions

*Question:* How many non-negative integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$?

- Give some examples of solutions.
- Characterize what solutions look like.
- A combinatorial object with a similar flavor is:

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In general, the number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = k$ is ________.

*Question:* How many **positive** integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$, where $x_4 \geq 3$?

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This leads to my favorite kind of proof:

*Definition:* A **combinatorial proof** of an identity $X = Y$ is a proof by counting (!). You find a set of objects that can be interpreted as a combinatorial interpretation of both the left hand side (LHS) and the right hand side (RHS) of the equation. As both sides of the equation count the same set of objects, they must be equal!

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- It is important to get the set of objects right.
- To do this, you must ask a good question: "In how many ways..."

# A Simple Combinatorial Proof

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Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

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*Question:* In how many ways can we choose from $n$ club members a committee of $k$ members with a chairperson?

*Answer 1:*

*Answer 2:*

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# Pascal's Identity

Example. Prove *Theorem* 2.2.1: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

**Combinatorial Proof:**

*Question:* In how many ways can we choose $k$ flavors of ice cream if $n$ different choices are available?

*Answer 1:*

*Answer 2:*

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

# Summing Binomial Coefficients

Example. Prove *Equation* (2.3): $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

**Analytic Proof:** ???

**Combinatorial Proof:**

*Question:* How many subsets of $\{1, 2, \ldots, n\}$ are there?

*Answer 1:* Condition on how many elements are in a subset.

*Answer 2:*

Because the two quantities count the same set of objects in two different ways, the two answers are equal. $\square$

—Worksheet—