mmooRREE COUNTING!

Question: In how many ways can we place k objects in n boxes?

Answer: It depends.

- What do the objects look like?
  - Do the objects all look the same?
- What do the boxes look like?
  - Do the boxes all look the same?
- Are there any restrictions?
  - Is there a size limit?
  - Must there be an object in each box?

Counting distributions

Definition: A **distribution** is an assignment of objects to recipients.

Certain counting problems can be revisited in this framework:

$$\left\{ \begin{array}{c} \text{Five-letter passwords} \\ \text{on } \{A, B, C, D, E, F, G\} \\ \end{array} \right\} \text{ correspond to } \left\{ \begin{array}{c} \text{Distributions of} \\ \underline{} \text{ distinct objects} \\ \text{into } \underline{} \text{ distinct boxes} \\ \end{array} \right\}$$

- What are candidates for objects, boxes?
- View as a function
- View as a distribution
- Find the restriction

THE CHART

Question: In how many ways can we place k objects in n boxes?

Distributions of		Restrictions on # objects received			
k **objects**	n **boxes**	none	≤ 1	≥ 1	$= 1$
distinct	distinct				
identical	distinct				
distinct	identical				
identical	identical				

Where do our known answers fit into the table? (Use function view)

- n^k: Objects distinct, Boxes distinct, no restriction.
- $(n)_k$: Objects distinct, Boxes distinct, ≤ 1 object per box.
- $n!$: Permutations. What about when $n \neq k$?
- $\binom{n}{k}$: Objects ________, Boxes _________, ___________________.
- $\left(\!\binom{n}{k}\!\right)$: Objects ________, Boxes _________, ___________________.

THE CHART

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identical	identical				

We can also fill in these answers:

- Objects identical, Boxes distinct, ≥ 1 object per box:
- Objects identical, Boxes distinct, $= 1$ object per box:

Pascal's triangle

Pascal's identity gives us the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
With initial conditions we can calculate $\binom{n}{k}$ for all n and k.
$\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all n.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1							1

Seq's in Pascal's triangle:

- $1, 2, 3, 4, 5, \ldots$ $\binom{n}{1}$
  ($a_n = n$) A000027
- $1, 3, 6, 10, 15, \ldots$ $\binom{n}{2}$
  triangular A000217
- $1, 4, 10, 20, 35, \ldots$ $\binom{n}{3}$
  tetrahedral A000292
- $1, 2, 6, 20, 70, \ldots$ $\binom{2n}{n}$
  centr. binom. A000984

Online Encyclopedia of Integer Sequences:

`http://www.research.att.com/~njas/sequences/`

Binomial Theorem

Theorem 2.2.2. Let n be a positive integer. For all x and y,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + y^n.$$

Rewrite in summation notation!
Determine the generic term $[\binom{n}{k}x \quad y \quad]$ and the bounds on k

$$(x+y)^n = \sum$$

- The entries of Pascal's triangle are the coefficients of terms in the expansion of $(x+y)^n$.

Proof. In the expansion of $(x+y)(x+y)\cdots(x+y)$, in how many ways can a term have the form $x^{n-k}y^k$?

From the n factors $(x+y)$, you must choose a "y" exactly k times. Therefore, $\binom{n}{k}$ ways. We recover the desired equation. □

Counting integral solutions

Question: How many non-negative integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$?

- Give some examples of solutions.
- Characterize what solutions look like.
- A combinatorial object with a similar flavor is:

In general, the number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = k$ is ________.

Question: How many **positive** integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$, where $x_4 \geq 3$?

What is a Combinatorial Proof?

Definition: A **combinatorial interpretation** of a numerical quantity is a set of combinatorial objects that is counted by the quantity.

Example. We can choose k objects out of n total objects in $\binom{n}{k}$ ways.

Use this fact "backwards" by interpreting an occurrence of $\binom{n}{k}$ as the number of ways to choose k objects out of n.

This leads to my favorite kind of proof:

Definition: A **combinatorial proof** of an identity $X = Y$ is a proof by counting (!). You find a set of objects that can be interpreted as a combinatorial interpretation of both the left hand side (LHS) and the right hand side (RHS) of the equation. As both sides of the equation count the same set of objects, they must be equal!

- It is important to get the set of objects right.
- To do this, you must ask a good question: "In how many ways..."

A Simple Combinatorial Proof

Example. Prove *Equation* (2.2): For $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$.
(We already know a bijective proof of this fact.)

Analytic Proof: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$

Combinatorial Proof:

Question: In how many ways can we adopt k of n cats available for adoption at the animal shelter?

Answer 1: Choose k of the n cats to adopt in $\binom{n}{k}$ ways.

Answer 2: Choose $n-k$ of the n cats to NOT adopt in $\binom{n}{n-k}$ ways.

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

Another Simple Combinatorial Proof

Example. Prove *Equation* (2.4): $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Analytic Proof:

Combinatorial Proof:

Question: In how many ways can we choose from n club members a committee of k members with a chairperson?

Answer 1:

Answer 2:

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

Pascal's Identity

Example. Prove *Theorem* 2.2.1: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Combinatorial Proof:

Question: In how many ways can we choose k flavors of ice cream if n different choices are available?

Answer 1:

Answer 2:

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

Summing Binomial Coefficients

Example. Prove *Equation* (2.3): $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Analytic Proof: ???

Combinatorial Proof:

Question: How many subsets of $\{1, 2, \ldots, n\}$ are there?

Answer 1: Condition on how many elements are in a subset.

Answer 2:

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

—Worksheet—