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In order to approach counting questions involving symmetry rigorously, we use the mathematical notion of *equivalence relation*.

Equivalence relations

Definition: An **equivalence relation** $\mathcal{E}$ on a set A satisfies the following properties:

- **Reflexive**: For all $a \in A$, $a \mathcal{E} a$.
- **Symmetric**: For all $a, b \in A$, if $a \mathcal{E} b$, then $b \mathcal{E} a$.
- **Transitive**: For all $a, b, c \in A$, if $a \mathcal{E} b$, and $b \mathcal{E} c$, then $a \mathcal{E} c$.

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- Our original question asks to count *equivalence classes* **(!)**.
- *Theorem 1.4.3.* If $a\mathcal{E}b$, then $\mathcal{E}(a) = \mathcal{E}(b)$.
- Every element of A is in *one* and *only one* equivalence class.
  - We say: "The equivalence classes of $\mathcal{E}$ partition A."

Equivalence classes partition A

Definition: A **partition** of a set S is a set of non-empty disjoint subsets of S whose union is S.

Example. Partitions of $S = \{*, \heartsuit, \clubsuit, ?\}$ include:

- $\big\{\{*, \heartsuit\}, \{?\}, \{\clubsuit\}\big\}$
- $\big\{\{\heartsuit, \clubsuit\}, \{*, ?\}\big\}$

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Key idea: (Thm 1.4.5) The set of equivalence classes of A partitions A.

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The equivalence principle: (p. 37) Let $\mathcal{E}$ be an equivalence relation on a finite set A. If every equivalence class has size C, then $\mathcal{E}$ has $|A|/C$ equivalence classes. (DIVISION!)

The Equivalence Principle

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Setup: Let A be the set of 10-lists, $(a_1, a_2, \ldots, a_9, a_{10}) = a \in A$.

This represents the pairings $\big\{\{a_1, a_2\}, \ldots, \{a_9, a_{10}\}\big\}$.

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Define two lists a and b to be equivalent if they give the same pairings.

[*For example,* $(3, 2, 9, 10, 1, 5, 8, 7, 4, 6) \equiv (2, 3, 9, 10, 1, 5, 6, 4, 8, 7)$.]

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We ask: How many different 10-lists are in the same equivalence class?
Answer:

By the equivalence principle,

Permutations of multisets

Example. How many different orderings are there of the letters in the word MISSISSIPPI?

Setup: If the letters were all distinguishable, we would have a permutation of 11 letters, $\{M, P, P, I, I, I, I, S, S, S, S\}$.

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- In how many ways can you position the S's?
- With S's placed, how many choices for the I's?
- With S's, I's placed, how many choices for the P's?
- With S's, I's, P's placed, how many choices for the M?

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Solution. The conjugacy classes correspond to ________________.