Introduction to Symmetry

Many combinatorial objects have a natural symmetry.

Example. In how many ways can we seat 4 people at a round table?

There are 4! permutations; however, each of _____ rotations gives the same order of guests. *Dividing* gives the _____ arrangements.

- In how many ways can we arrange 10 people into five pairs?
- In how many ways can we k-color the vertices of a square?

In order to approach counting questions involving symmetry rigorously, we use the mathematical notion of *equivalence relation*.

Equivalence relations

Definition: An **equivalence relation** $\mathcal{E}$ on a set A satisfies the following properties:

- **Reflexive**: For all $a \in A$, $a\mathcal{E}a$.
- **Symmetric**: For all $a, b \in A$, if $a\mathcal{E}b$, then $b\mathcal{E}a$.
- **Transitive**: For all $a, b, c \in A$, if $a\mathcal{E}b$, and $b\mathcal{E}c$, then $a\mathcal{E}c$.

Example. When sitting four people at a round table, let A be all 4-permutations. We say $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$ are "equivalent" ($a\mathcal{E}b$) if they are rotations of each other.

Verify that $\mathcal{E}$ is an equivalence relation.

- It is reflexive because:
- It is symmetric because:
- It is transitive because:

Equivalence classes

It is natural to investigate the set of all elements related to a:

Definition: The **equivalence class containing** a is the set

$$\mathcal{E}(a) = \{x \in A : x\mathcal{E}a\}.$$

Class 1: { (1,2,3,4) , (2,3,4,1) , (3,4,1,2) , (4,1,2,3) }
Class 2: { (1,2,4,3) , (2,4,3,1) , (4,3,1,2) , (3,1,2,4) }
Class 3: { (1,3,2,4) , (3,2,4,1) , (2,4,1,3) , (4,1,3,2) }
Class 4: { (1,3,4,2) , (3,4,2,1) , (4,2,1,3) , (2,1,3,4) }
Class 5: { (1,4,2,3) , (4,2,3,1) , (2,3,1,4) , (3,1,4,2) }
Class 6: { (1,4,3,2) , (4,3,2,1) , (3,2,1,4) , (2,1,4,3) }

- Our original question asks to count *equivalence classes* **(!)**.
- *Theorem 1.4.3.* If $a\mathcal{E}b$, then $\mathcal{E}(a) = \mathcal{E}(b)$.
- Every element of A is in *one* and *only one* equivalence class.
  - We say: "The equivalence classes of $\mathcal{E}$ partition A."

Equivalence classes partition A

Definition: A **partition** of a set S is a set of non-empty disjoint subsets of S whose union is S.

Example. Partitions of $S = \{*, \heartsuit, \clubsuit, ?\}$ include:

- $\{\{*, \heartsuit\}, \{?\}, \{\clubsuit\}\}$
- $\{\{\heartsuit, \clubsuit\}, \{*, ?\}\}$

Every element is in some subset and no element is in multiple subsets.

Key idea: (Thm 1.4.5) The set of equivalence classes of A partitions A.

- Every equivalence class is non-empty.
- Every element of A is in *one* and *only one* equivalence class.

The equivalence principle: (p. 37) Let $\mathcal{E}$ be an equivalence relation on a finite set A. If every equivalence class has size C, then $\mathcal{E}$ has $|A|/C$ equivalence classes. (DIVISION!)

The Equivalence Principle

Example. In how many ways can we arrange 10 people into five pairs?

Setup: Let A be the set of 10-lists, $(a_1, a_2, \ldots, a_9, a_{10}) = a \in A$.

This represents the pairings $\{\{a_1, a_2\}, \ldots, \{a_9, a_{10}\}\}$.

Define two lists a and b to be equivalent if they give the same pairings.

[*For example,* $(3, 2, 9, 10, 1, 5, 8, 7, 4, 6) \equiv (2, 3, 9, 10, 1, 5, 6, 4, 8, 7)$.]

(Why is this an equivalence relation?)

We ask: How many different 10-lists are in the same equivalence class?

Answer:

By the equivalence principle,

Permutations of multisets

Example. How many different orderings are there of the letters in the word MISSISSIPPI?

Setup: If the letters were all distinguishable, we would have a permutation of 11 letters, $\{M, P, P, I, I, I, I, S, S, S, S\}$.

Define $a \mathcal{E} b$ if a and b are the same word when color is ignored. (Is this an equivalence relation?)

Question: How many words are in the same equivalence class?

Alternatively, count directly.

- In how many ways can you position the S's?
- With S's placed, how many choices for the I's?
- With S's, I's placed, how many choices for the P's?
- With S's, I's, P's placed, how many choices for the M?

Blah Blah Blah

Careful: Conjugacy classes might not be of equal size.

Example. Let A be the subsets of [4]. Define $S\mathcal{E}T$ when $|S| = |T|$. Determine the number of conjugacy classes of $\mathcal{E}$.

Solution. (NOT) We know that $\mathcal{E}(\{1\}) = \big\{\{1\},\{2\},\{3\},\{4\}\big\}$, of size 4. Since $|A| = 24$, there are $\frac{24}{4} = 6$ conjugacy classes.

Solution. The conjugacy classes correspond to __________________.