Course Notes

Combinatorics, Fall 2012

Queens College, Math 636

Prof. Christopher Hanusa

On the web: `http://people.qc.cuny.edu/faculty/christopher.hanusa/courses/636fa12/`

Reference List

The following are books that I recommend to complement this course. I have asked that they be placed *on reserve* in the library.

Benjamin and Quinn. *Proofs that really count.*
Bóna. *A walk through combinatorics.*
Brualdi. *Introductory combinatorics.*
Graham, Knuth, and Patashnik. *Concrete mathematics.*
Mazur. *Combinatorics: A guided tour*
van Lint and Wilson. *A course in combinatorics.*

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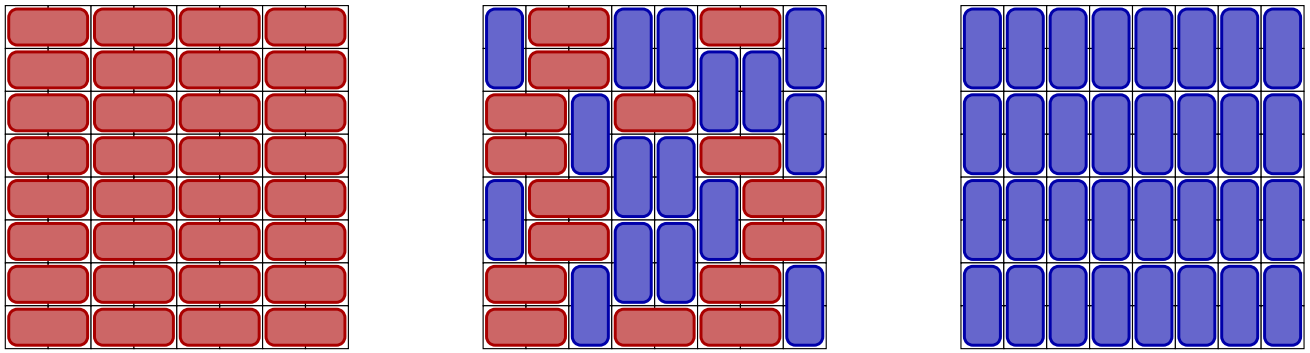

A **domino tiling** is a placement of dominoes on a region, where

- Each domino covers two squares.
- The dominoes cover the whole region and do not overlap.

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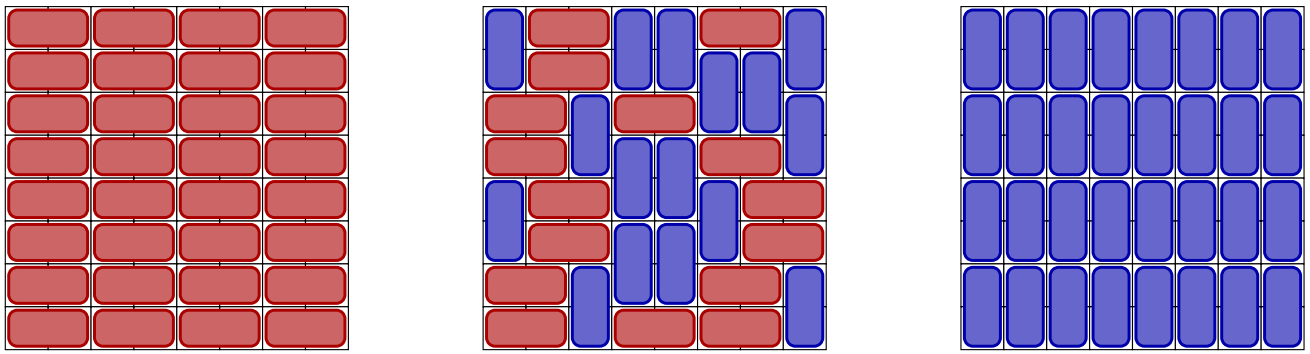

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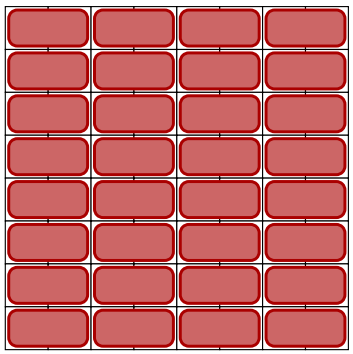
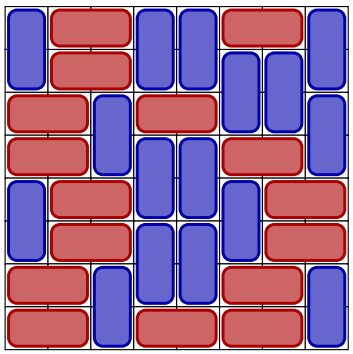
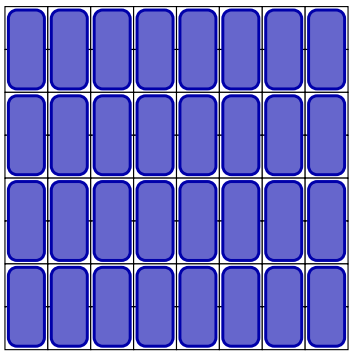

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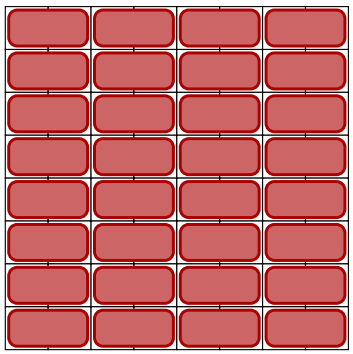
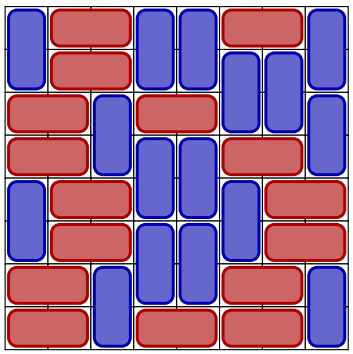
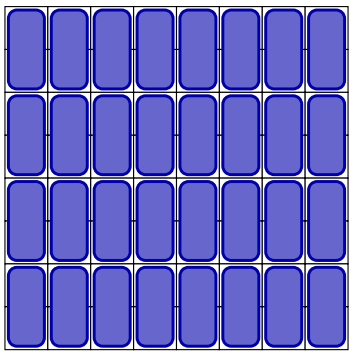

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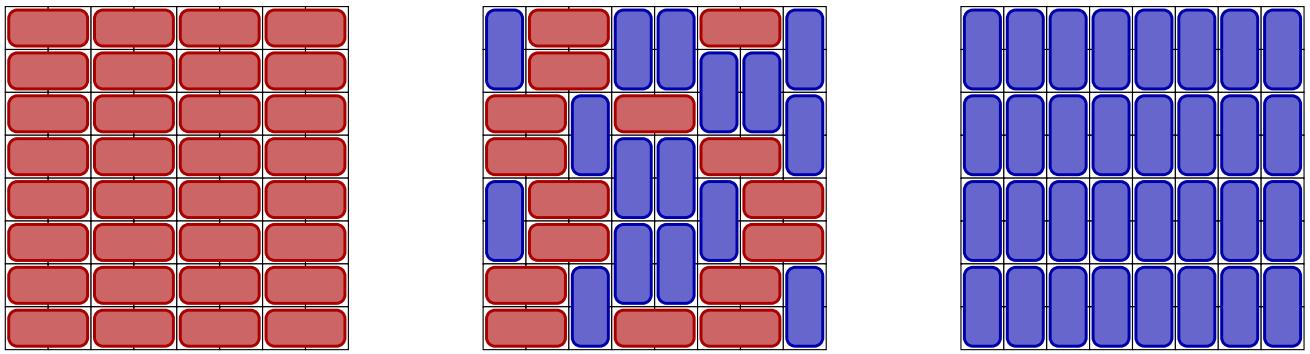

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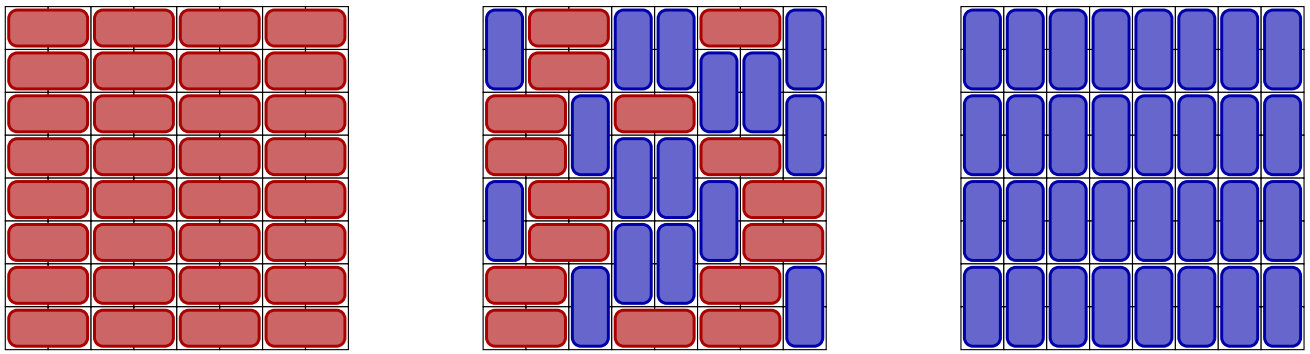

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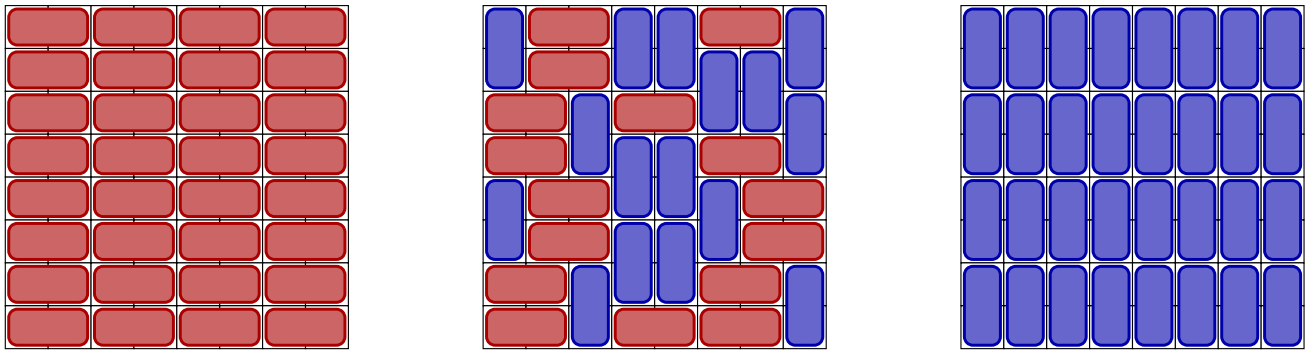

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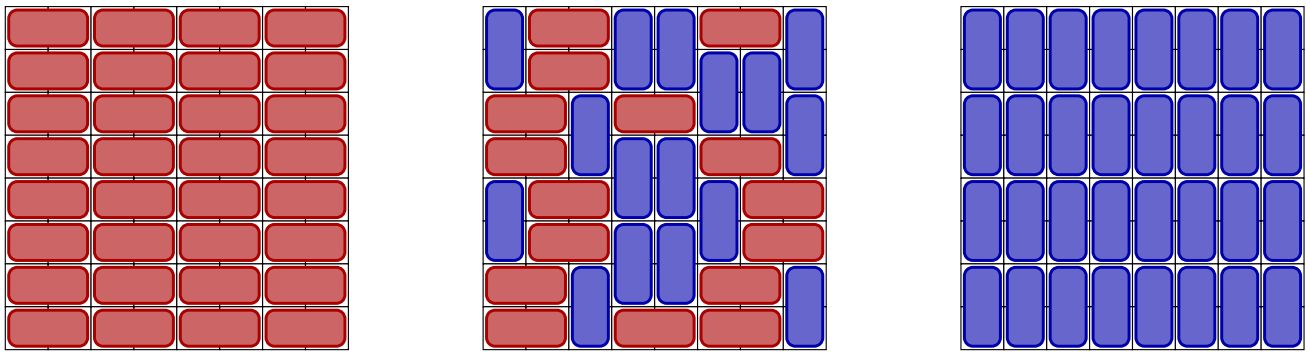

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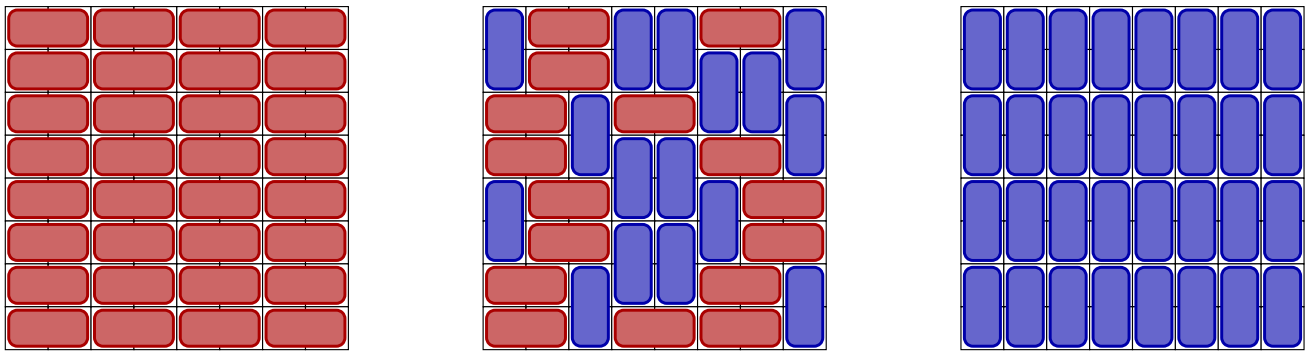

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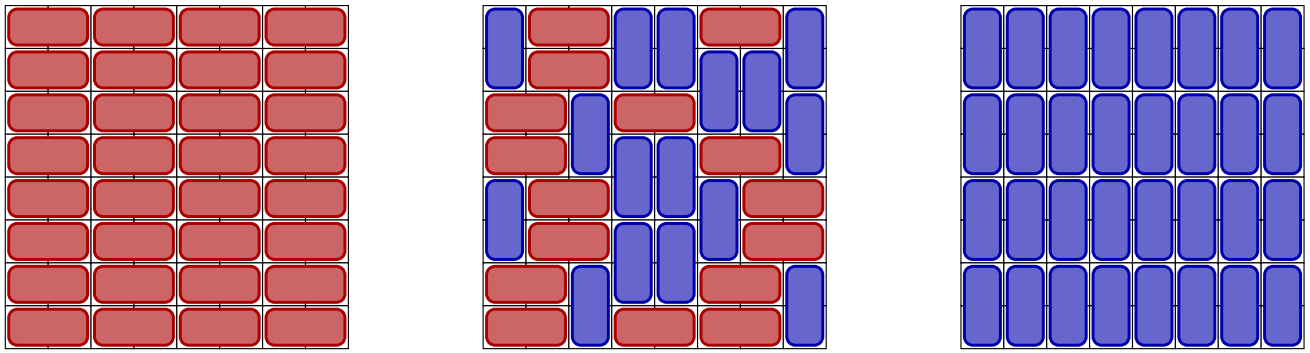

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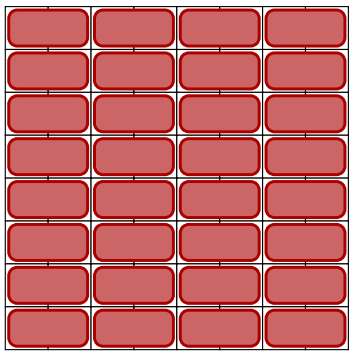
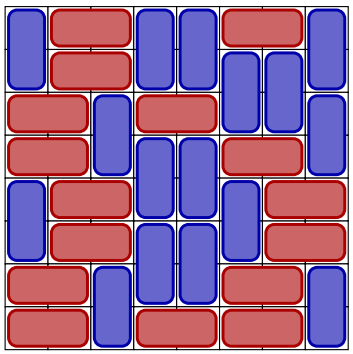
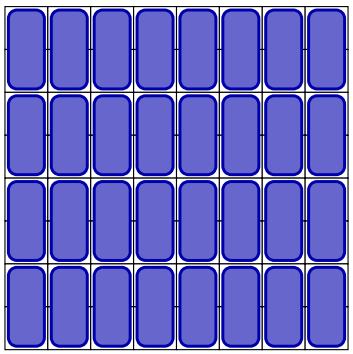

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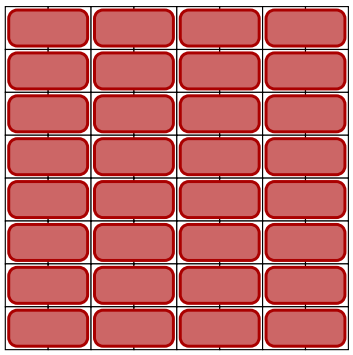
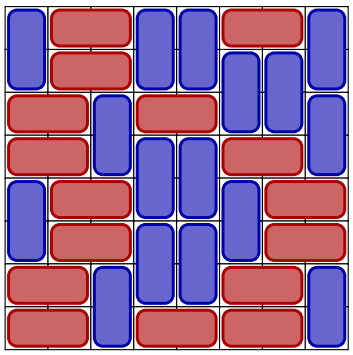
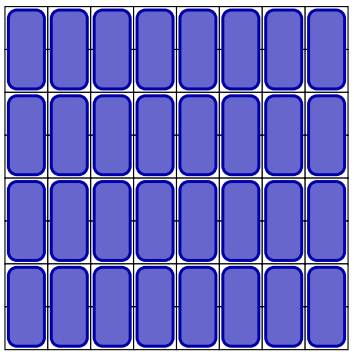

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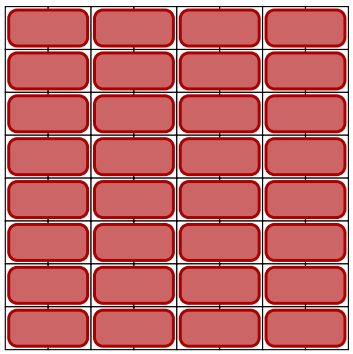
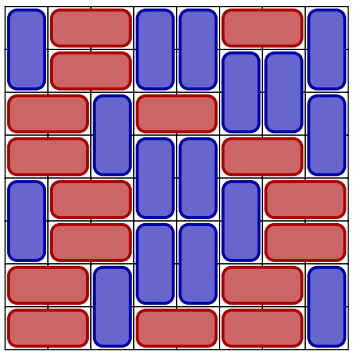
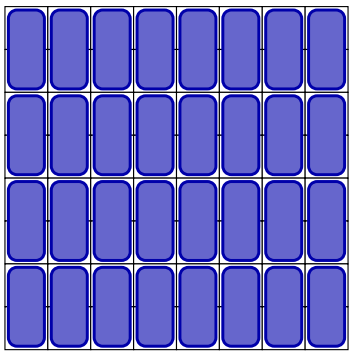

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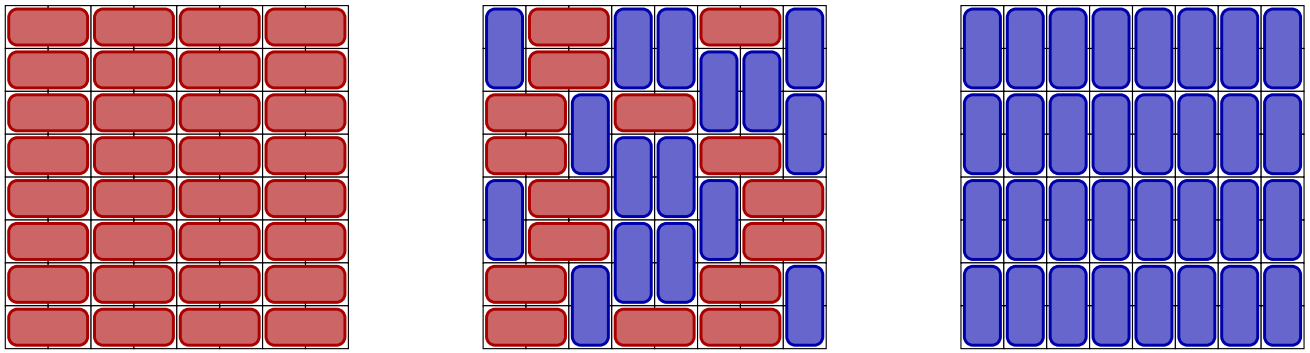

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How to determine the "answer"?

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Answer: If m and n are both even, then we have the **formula** (!):

$$\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right).$$

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Given some discrete objects, what properties and structures do they have?

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(Requires many proofs.) (Uses a different kind of reasoning!)

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All homeworks online; first one due next Thursday.

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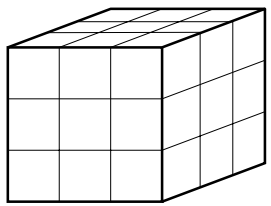
Cut a 3×3 cube into twenty-seven 1×1 cubes using as few cuts as possible. (Rearrangements are allowed.)



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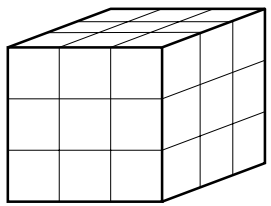

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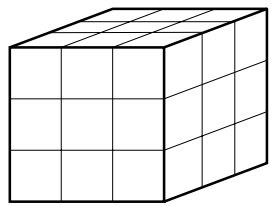

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Conjecture: 6 is the minimum possible number of cuts.
Proof:

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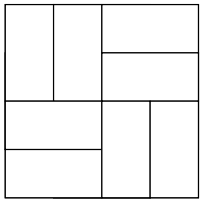

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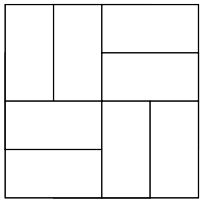

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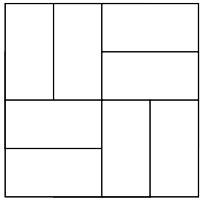

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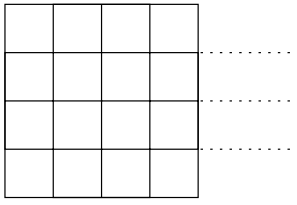

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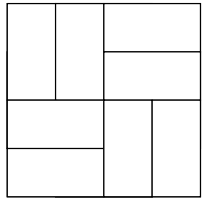

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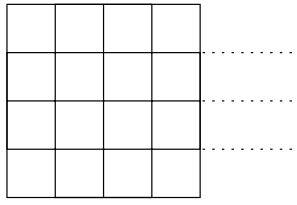

Every vertical domino must intersect exactly one of these separators; we can count the number of vertical dominoes by adding $x_1 + x_2 + x_3$.

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  - Similarly, there are at least six horizontal dominoes
- However, a 4×4 chessboard can only hold 8 dominoes, a contradiction!

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  - Not coverable by dominoes!
- Suppose there exists a 4×4 domino tiling with no fault line.
  - Therefore, x_1, x_2, and x_3 are all positive (and ≥ 2)
  - And there must be at least six vertical dominoes.
  - Similarly, there are at least six horizontal dominoes
- However, a 4×4 chessboard can only hold 8 dominoes, a contradiction!

Therefore, it is impossible for a 4×4 domino tiling to have no fault lines.

Numbers are everywhere

Arrange yourselves into groups of four or six people,
With people you don't know.

- Introduce yourself. (your name, where you are from)
- What brought you to this class?
- Numbers are everywhere.
  What is a number that you identify with?

Four Counting Questions (p. 2)

Here are four counting questions.

Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Q3. How many Pick-6 lottery tickets are there? (Choose six numbers between 1–40.)

Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Think Write Pair Share: Order these from smallest to largest.

_____ _____ _____ _____

The game of Nim

Here are the rules of the two-player game Nim:

1. The game starts with two piles of counters.
2. Alternating play, each player removes some number of counters from **either** pile.
3. The player who removes the last counter wins.

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Let's play!

- First, get a feel for the game. Try starting with initial piles of (4,6), (5,5), (3,10), and (7,8).
- Next, start to develop some strategies for winning.
- Finally, determine conditions under which the first player will always win if she plays optimally, and similarly for the second player.

If you finish this before time is up, try playing Nim with three or more initial piles.