Course Notes

Combinatorics, Fall 2012

Queens College, Math 636

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On the web: http://people.qc.cuny.edu/faculty/christopher.hanusa/courses/636fa12/

Reference List

The following are books that I recommend to complement this course. I have asked that they be placed *on reserve* in the library.

Benjamin and Quinn. *Proofs that really count.*
Bóna. *A walk through combinatorics.*
Brualdi. *Introductory combinatorics.*
Graham, Knuth, and Patashnik. *Concrete mathematics.*
Mazur. *Combinatorics: A guided tour*
van Lint and Wilson. *A course in combinatorics.*

What is combinatorics?

In this class: Learn how to count . . . **better**.

Question: How many domino tilings are there of an 8×8 chessboard?

A **domino tiling** is a placement of dominoes on a region, where

- Each domino covers two squares.
- The dominoes cover the whole region and do not overlap.

Domino tilings

How to determine the “answer”?

- Convert the chessboard into a combinatorial structure (a graph).
- Represent the graph numerically as a matrix.
- Take the determinant of this matrix.
- Use the structure of the matrices to determine their eigenvalues.

Question: How many domino tilings are there of an $m \times n$ board?

Answer: If m and n are both even, then we have the **formula** (!):

$$\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1} \right).$$

Combinatorial questions

Given some discrete objects, what properties and structures do they have?

- Can we count the arrangements?
  - **Count** means give a *number*.
- Can we enumerate the arrangements?
  - **Enumerate** means give a *description* or *list*.
- Do any arrangements have a certain property?
  - This is an **existence** question.
- Can we construct arrangements having some property?
  - We need to find a method of **construction**.
- Does there exist a "best" arrangement?
  - **Prove optimality**.

(Requires many proofs.) (Uses a different kind of reasoning!)

To do well in this class:

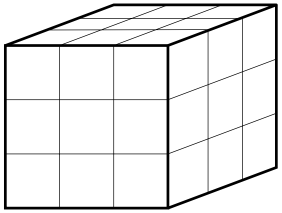
- **Come to class prepared.**
  - Print out and read over course notes.
  - Read sections before class.
- **Form good study groups.**
  - Discuss homework and classwork.
  - Bounce proof ideas around.
  - You will depend on this group.
- **Put in the time.**
  - Three credits = (at least) nine hours / week out of class.
  - Homework stresses key concepts from class; learning takes time.
- **Stay in contact.**
  - If you are confused, ask questions (in class and out).
  - Don't fall behind in coursework or project.
  - I need to understand your concerns.

All homeworks online; first one due next Thursday.

Cutting a cube

In this class: Learn how to count . . . **better**.

Cut a 3×3 cube into twenty-seven 1×1 cubes using as few cuts as possible. (Rearrangements are allowed.)

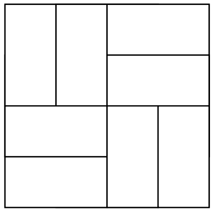


What is the simplest (most obvious) answer? ____________________

Can you do better?

Conjecture: ___ is the minimum possible number of cuts.
Proof:

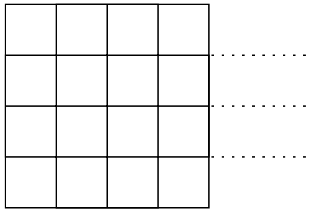
Fault Lines in Domino Tilings



Question: Which 4×4 domino tilings have a *fault line*?

Conjecture: **Every** domino tiling of a 4×4 board has a fault line.

Proof: Define x_1, x_2, and x_3 to be the number of dominoes crossing the first, second, and third separators, respectively:



Every vertical domino must intersect exactly one of these separators; we can count the number of vertical dominoes by adding $x_1 + x_2 + x_3$.

Fault Lines in Domino Tilings

- x_1, x_2, and x_3 cannot be odd.
  - The number of squares above the *odd* separator would be odd.
  - Not coverable by dominoes!
- Suppose there exists a 4×4 domino tiling with no fault line.
  - Therefore, x_1, x_2, and x_3 are all positive (and ≥ 2)
  - And there must be at least six vertical dominoes.
  - Similarly, there are at least six horizontal dominoes
- However, a 4×4 chessboard can only hold 8 dominoes, a contradiction!

Therefore, it is impossible for a 4×4 domino tiling to have no fault lines.

Numbers are everywhere

Arrange yourselves into groups of four or six people,
With people you don't know.

- Introduce yourself. (your name, where you are from)
- What brought you to this class?
- Numbers are everywhere.
  What is a number that you identify with?

Four Counting Questions (p. 2)

Here are four counting questions.

Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Q3. How many Pick-6 lottery tickets are there? (Choose six numbers between 1–40.)

Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Think Write Pair Share: Order these from smallest to largest.

______ ______ ______ ______

The game of Nim

Here are the rules of the two-player game Nim:

1. The game starts with two piles of counters.
2. Alternating play, each player removes some number of counters from **either** pile.
3. The player who removes the last counter wins.

Let's play!

- First, get a feel for the game. Try starting with initial piles of (4,6), (5,5), (3,10), and (7,8).
- Next, start to develop some strategies for winning.
- Finally, determine conditions under which the first player will always win if she plays optimally, and similarly for the second player.

If you finish this before time is up, try playing Nim with three or more initial piles.