Combinatorial Proof Practice

Prove: $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ for $0 \le k \le m \le n$.

Prove: $1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$.

Prove: $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Small hints

(1) Choose a subcommittee.
(2) Form a committee with a chairperson.
(3) Write $\binom{n}{k}^2$ as $\binom{n}{k}\binom{n}{n-k}$.
Break your set of size $2n$ into two smaller sets. (Color them blue and red, for example.)

Larger-in-use, smaller-in-size hints:

(1) How many ways are there to choose a subcommittee of size k from a committee of size m?
(2) Given a committee of size k, in how many ways are there to choose a chairperson of the committee?
(3) If you take your set of size $2n$ and color n elements blue and n elements red, and then choose n elements from the set of size $2n$, how might those chosen elements break down with respect to the colors?
What happens if k of them are blue?