The sum principle

Often it makes sense to break down your counting problem into smaller, disjoint, and easier-to-count sub-problems.

Example. How many integers from 1 to 999999 are palindromes?

Answer: Condition on how many digits.

- Length 1:
- Length 2:
- Length 3:
- Length 4:
- Length 5,6:
- **Total**:

★ Every palindrome between 1 and 999999 is counted once.

This illustrates the **sum principle**:

Suppose the objects to be counted can be broken into k disjoint and exhaustive cases. If there are n_j objects in case j, then there are $n_1 + n_2 + \cdots + n_k$ objects in all.

Counting pitfalls

When counting, there are two common pitfalls:

- Undercounting
  - Often, forgetting cases when applying the sum principle.
  - **Ask:** Did I miss something?
- Overcounting
  - Often, misapplying the product principle.
  - **Ask:** Do cases need to be counted in different ways?
  - **Ask:** Does the same object appear in multiple ways?

Common example: A deck of cards.

There are four suits: Diamond $\diamondsuit$, Heart $\heartsuit$, Club $\clubsuit$, Spade $\spadesuit$.
Each has 13 cards: Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2.

Example. Suppose you are dealt two diamonds between 2 and 10.
In how many ways can the product be even?

Overcounting

Example. In Blackjack you are dealt 2 cards: 1 face-up, 1 face-down. In how many ways can the face-down card be an Ace and the face-up card be a Heart ♡?

Answer: There are __ aces, so there are ___ choices for the down card. There are __ hearts, so there are ____ choices for the up card. By the product principle, there are 52 ways in all.

Except:

Remember to ask: Do cases need to be counted in different ways?

Overcounting

Example. How many 4-lists taken from [9] have at least one pair of adjacent elements equal?

Examples: 1114, 1229, 5555 Non-examples: 1231, 9898.

Strategy:

1. Choose where the adjacent equal elements are. (___ ways)
2. Choose which number they are. (___ ways)
3. Choose the numbers for the remaining elements. (___ ways)

By the product principle, there are ________ ways in all.

Except:

Remember to ask: Does the same object appear in multiple ways?

Counting the complement

Q1: How many 4-lists taken from [9] have **at least one** pair of adjacent elements equal?

—Compare this to—

Q2: How many 4-lists taken from [9] have **no** pairs of adjacent elements equal?

What can we say about:

Q1: **Q2:** **Together:**

Q3:

Strategy: It is sometimes easier to **count the complement**.

Answer to Q3:

Answer to Q2:

Answer to Q1:

Poker hands

Example. When playing five-card poker, what is the probability that you are dealt a full house?

[*Three cards of one type and two cards of another type.*] 5 5 5 K K

Game plan:

- Count the total number of hands.

- Count the number of possible full houses. **# of ways**
  - Choose the denomination of the three-of-a-kind.
  - Choose which three suits they are in.
  - Choose the denomination of the pair.
  - Choose which two suits they are in.
  - Apply the multiplication principle. **Total:**

- Divide to find the probability.