Course Notes

Combinatorics, Fall 2011

Queens College, Math 636

Prof. Christopher Hanusa

On the web: `http://people.qc.cuny.edu/faculty/christopher.hanusa/courses/636fa11/`

Reference List

The following are books that I recommend to complement this course. I have asked that they be placed *on reserve* in the library.

Benjamin and Quinn. *Proofs that really count.*
Bóna. *A walk through combinatorics.*
Brualdi. *Introductory combinatorics.*
Graham, Knuth, and Patashnik. *Concrete mathematics.*
Mazur. *Combinatorics: A guided tour*
van Lint and Wilson. *A course in combinatorics.*

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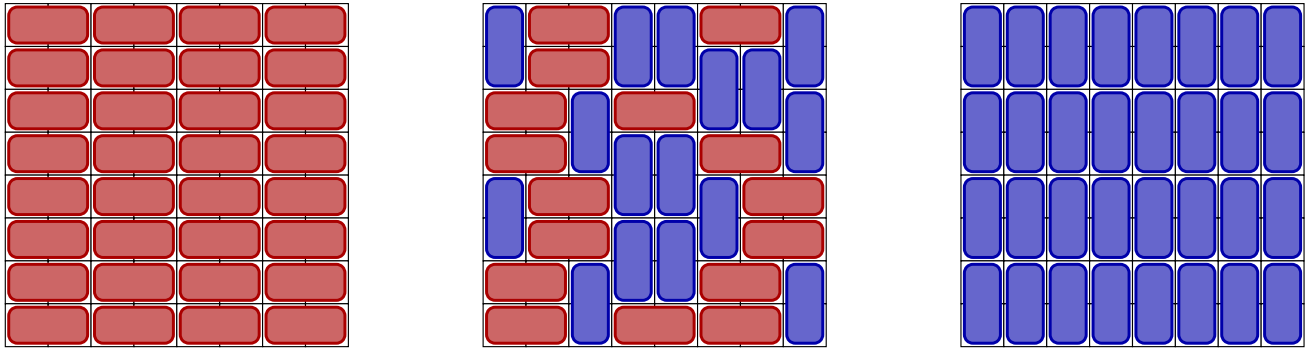

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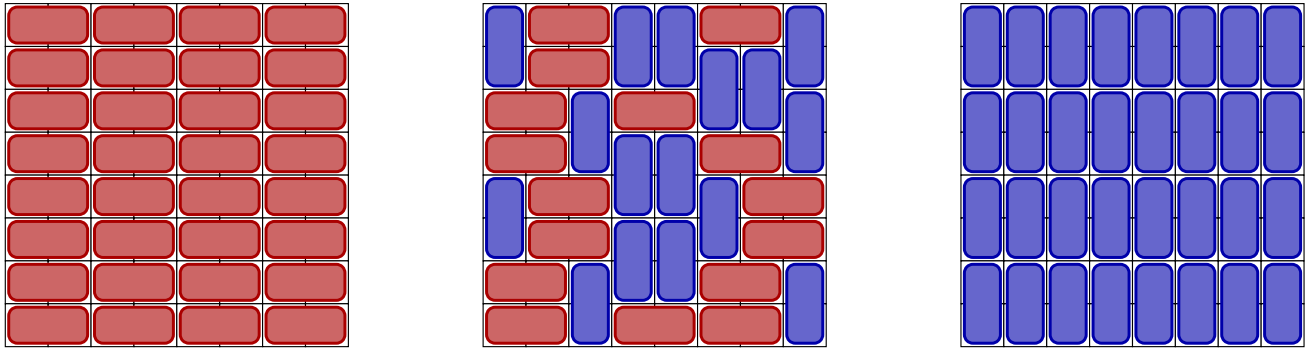

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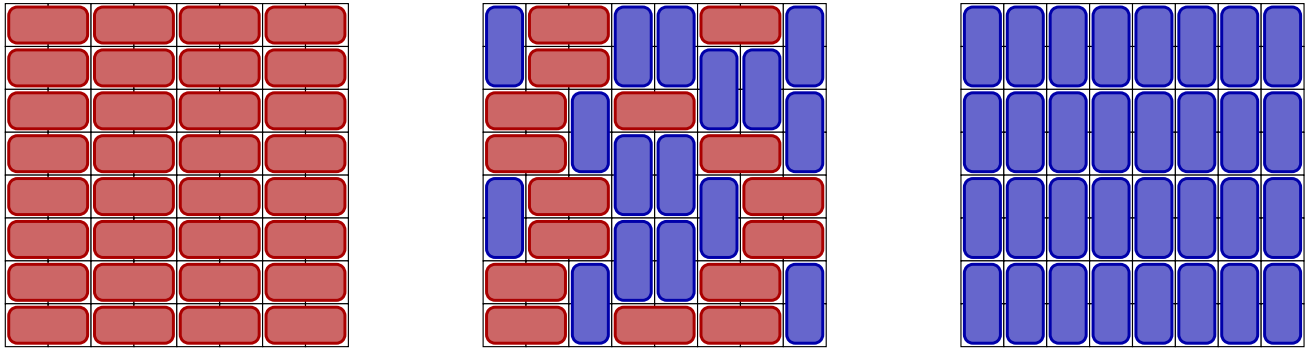

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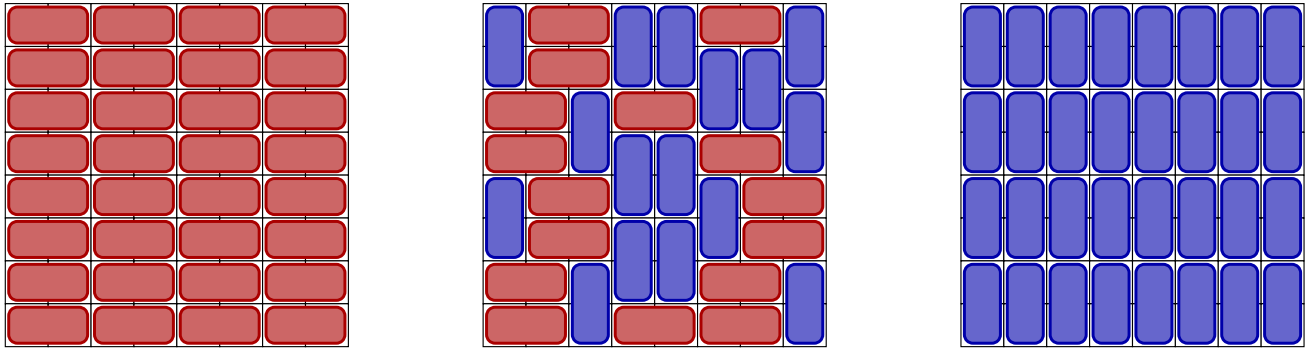

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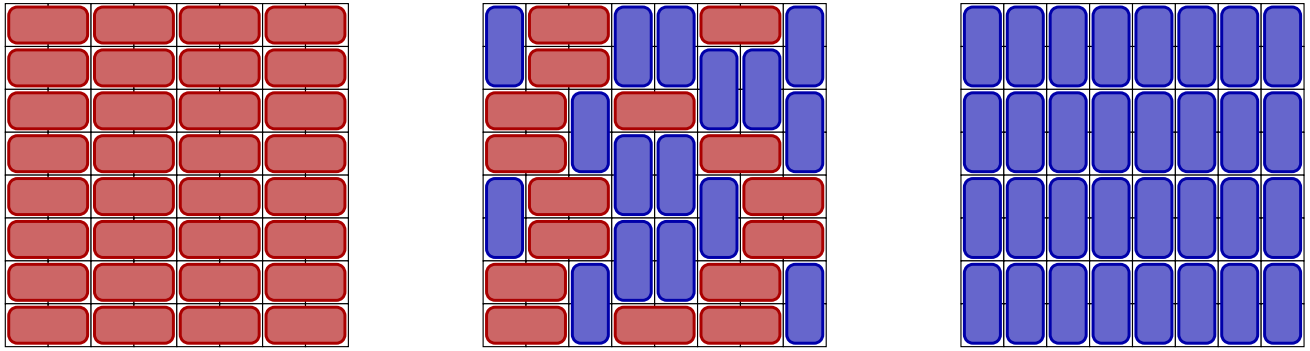

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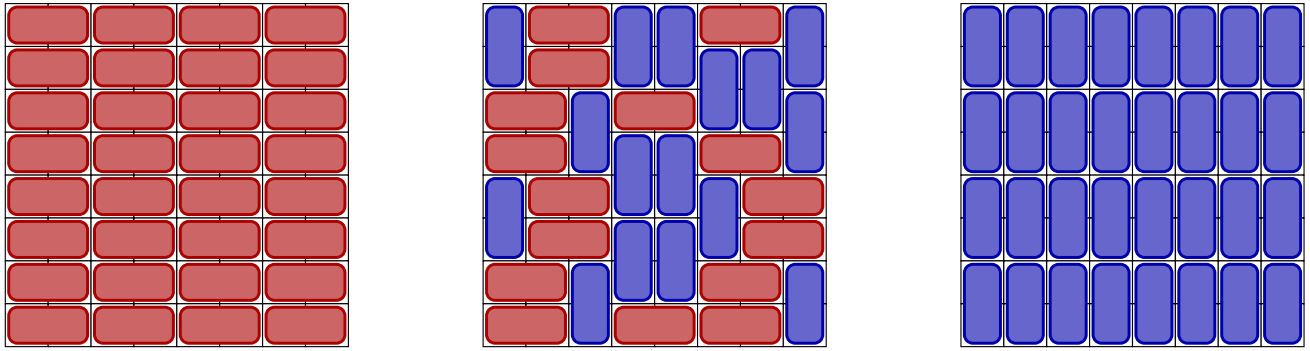

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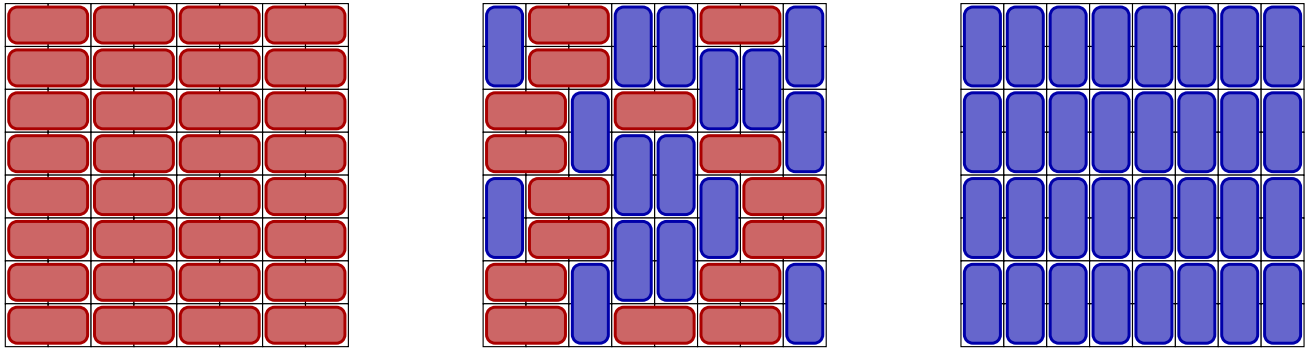

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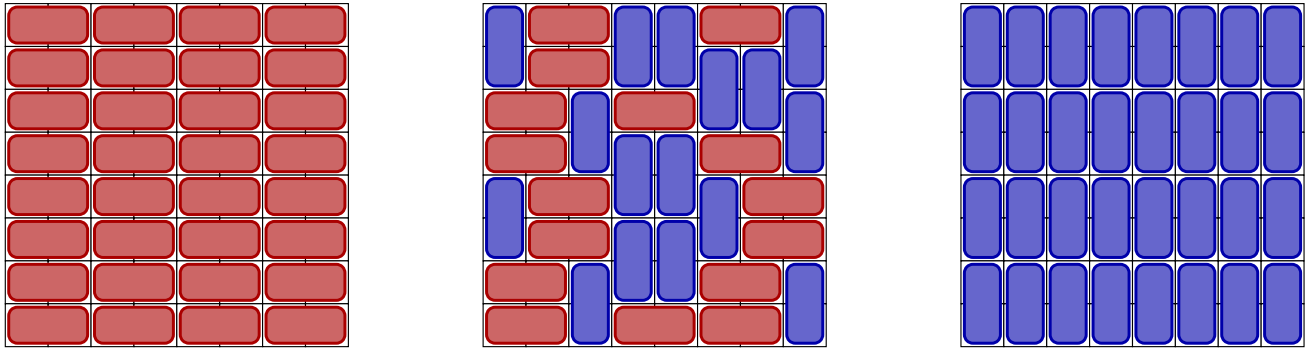

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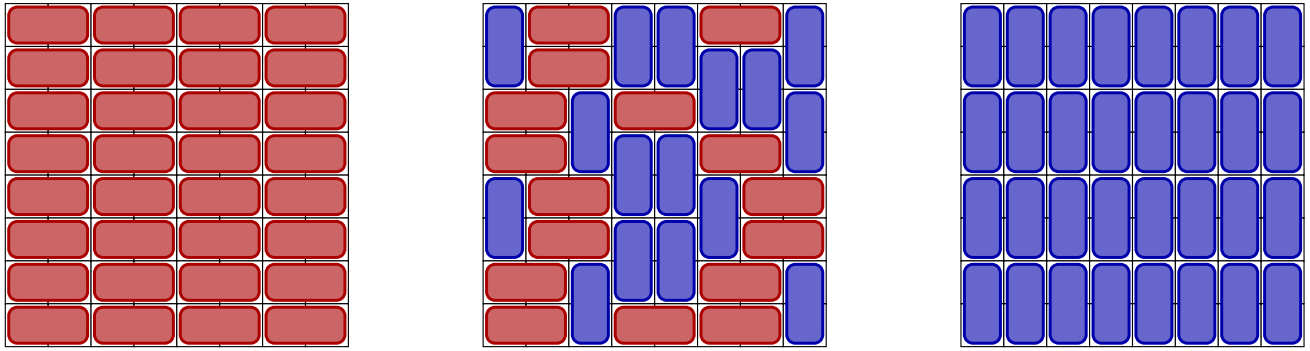

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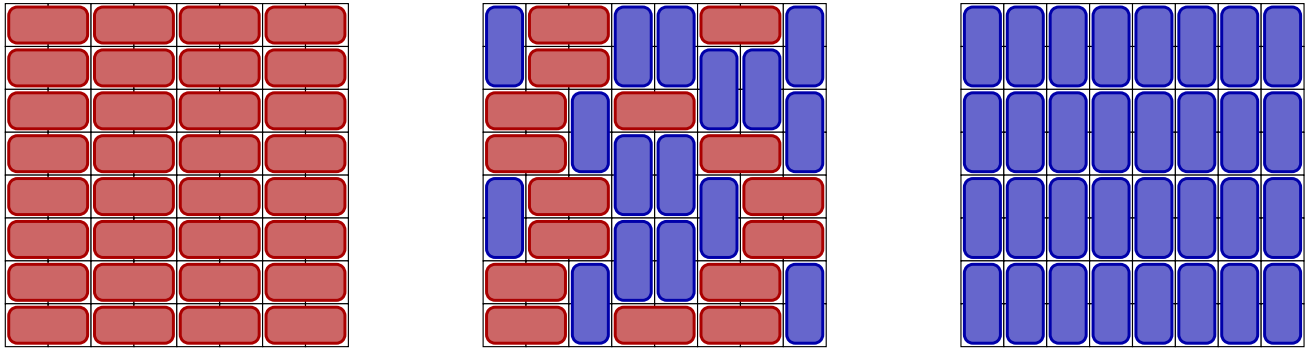

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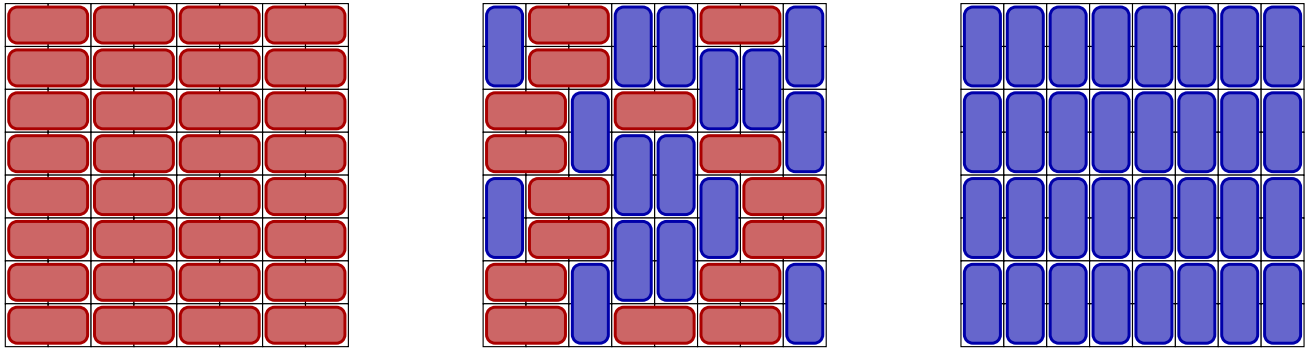

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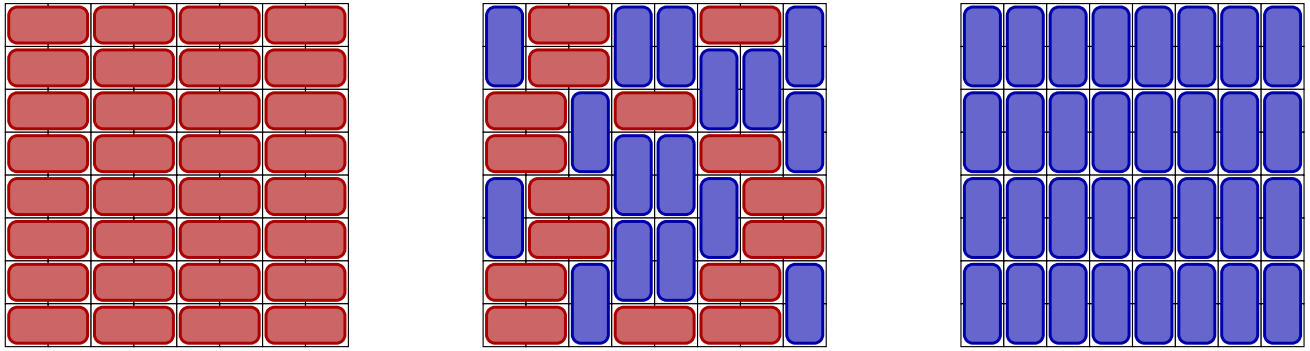

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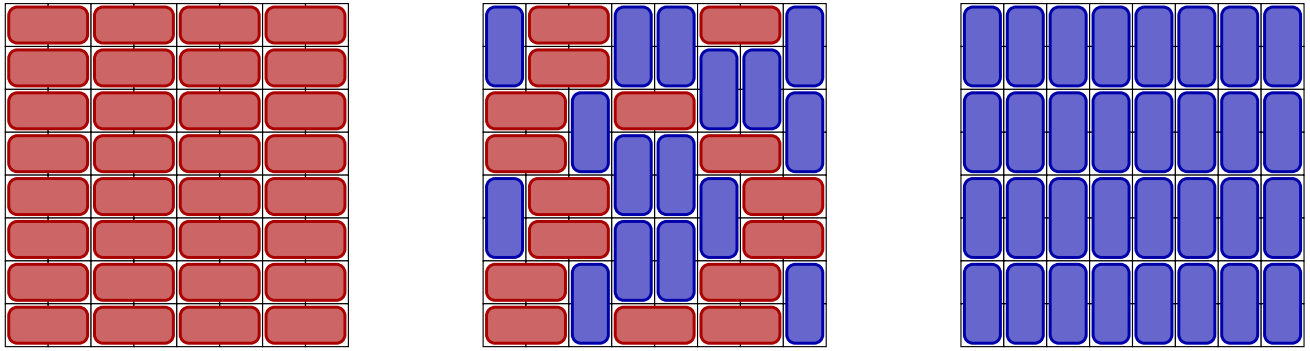

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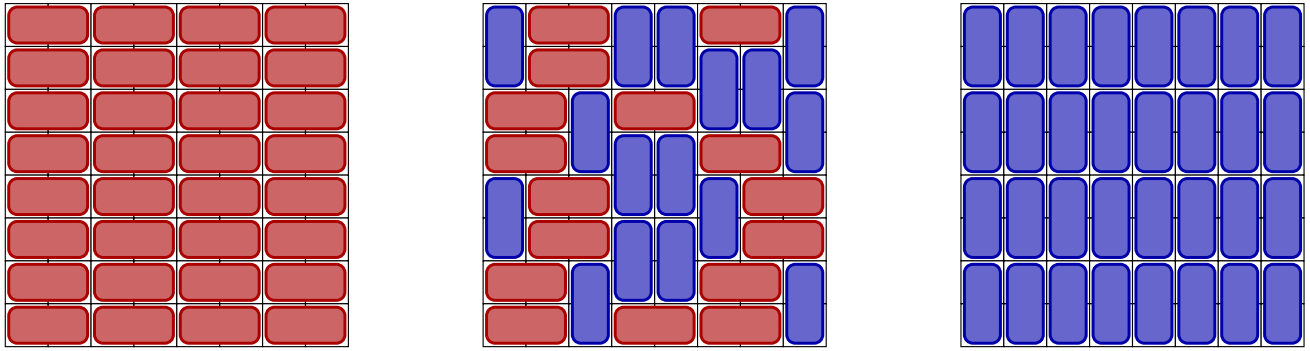

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How to determine the "answer"?

- Convert the chessboard into a combinatorial structure (a graph).
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Answer: If m and n are both even, then we have the **formula** (!):

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(Requires many proofs.) (Uses a different kind of reasoning!)

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All homeworks online; first one due next Wednesday.

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Think Write Pair Share: Order these from smallest to largest.

_____ _____ _____ _____

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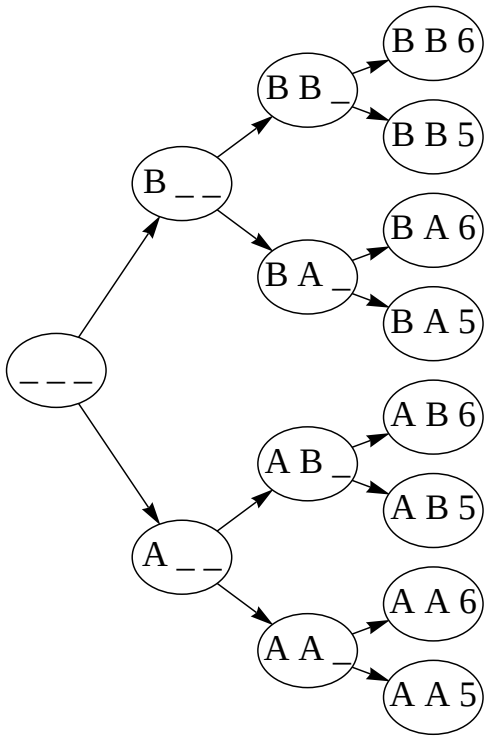

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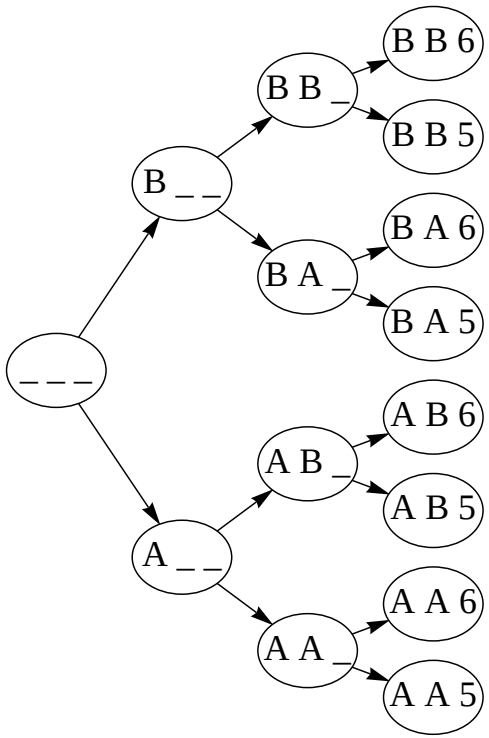

Alternatively: Notice two *independent* choices for each character. Multiply $2 \cdot 2 \cdot 2 = 8$.

The Product Principle

This illustrates:

The product principle: When counting lists $(l_1, l_2, \ldots, l_k)$,

IF there are c_1 choices for entry l_1, each leading to a different list.

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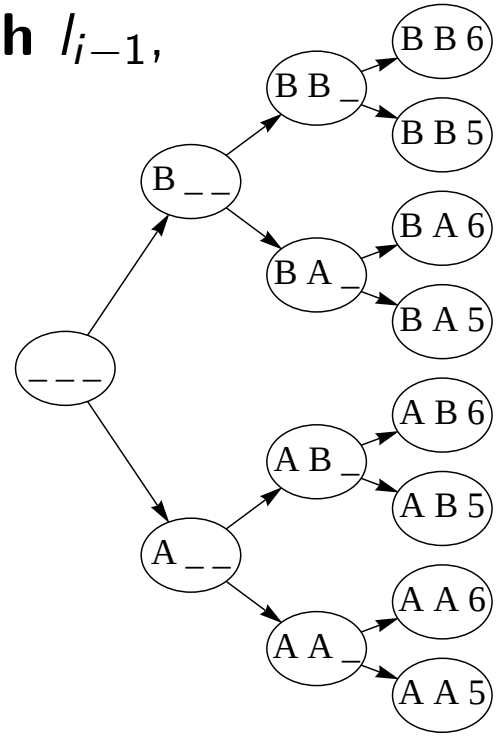

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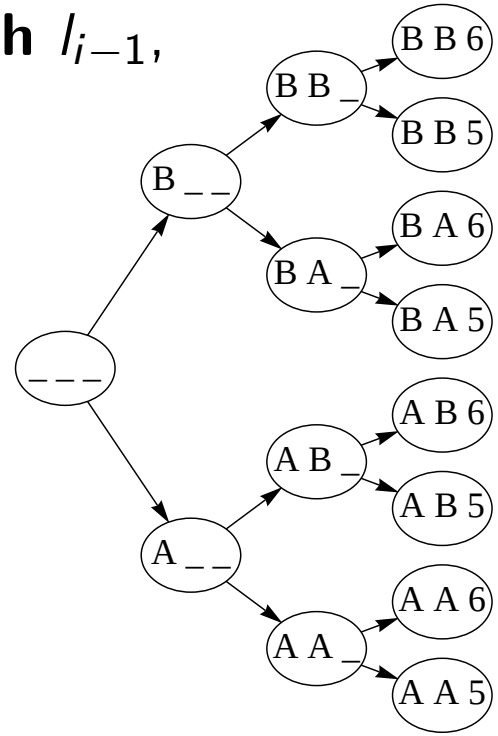

Caution: The product principle seems simple, but we must be careful when we use it.

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In general, the number of words of length k that can be made from an alphabet of length n and <u>where repetition is allowed</u> is n^k

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This number also counts _____.

Application: Counting Subsets

Example. How many subsets of a set $S = \{s_1, s_2, \ldots, s_n\}$ are there?

Strategy: "Try small problems, see a pattern."

- $n = 0$: $S = \emptyset \rightsquigarrow \{\emptyset\}$, size 1.
- $n = 1$: $S = \{s_1\} \rightsquigarrow \{\emptyset, \{s_1\}\}$, size 2.
- $n = 2$: $S = \{s_1, s_2\} \rightsquigarrow \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}\}$, size 4.
- $n = 3$: $S = \{s_1, s_2, s_3\} \rightsquigarrow \left\{\begin{matrix} \emptyset, & \{s_1\}, & \{s_2\}, & \{s_1, s_2\}, \\ \{s_3\}, & \{s_1, s_3\}, & \{s_2, s_3\}, & \{s_1, s_2, s_3\} \end{matrix}\right\}$, 8.

It appears that the number of subsets of S is _____. (notation)

This number also counts _____.

We can label the subsets by whether or not they contain s_i.

For example, for $n = 3$, we label the subsets $\left\{\begin{matrix} 000, 100, 010, 011, \\ 100, 101, 011, 111 \end{matrix}\right\}$

Permutations

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Answer: The number of choices for each lineup spot are:

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Question: How many 8-character passwords are there using A–Z, a–z, 0–9, containing no repeated character?

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- That is, the number of k-permutations of an n-set is $(n)_k$.
- Special case: For n-permutations of an n-set: $n!$.

Notation

Some quantities appear frequently, so we use shorthand notation:

- $[n] := \{1, 2, \ldots, n\}$
- $2^S :=$ set of all subsets of S
- $n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$
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- The individual objects we are counting are unordered.
  They are <u>subsets</u>, not lists.

A formula for $\binom{n}{k}$

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Since we counted the same quantity twice, they must be equal!

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[In other words, *list them all* or *completely describe the list*.]

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Answer:

How would you describe a k-multisubset of $[n]$?

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Question: How many k-multisets are there from an n-set?

— is the same as —

Question: How many ways are there to place k indistinguishable balls into n distinguishable bins?

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$$\{a^2, b^0, c^3, d^1\} \quad \begin{array}{l} n = 4 \\ k = 6 \end{array}$$

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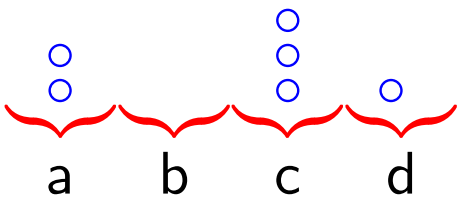

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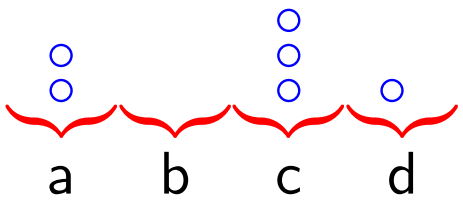

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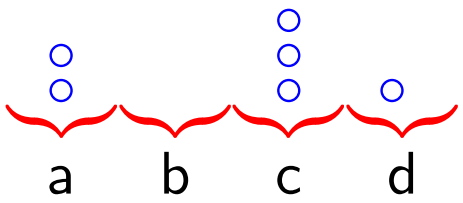

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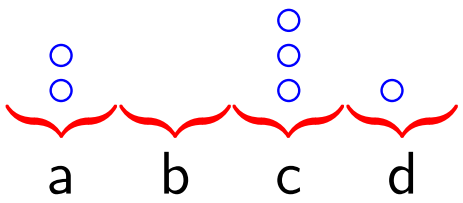

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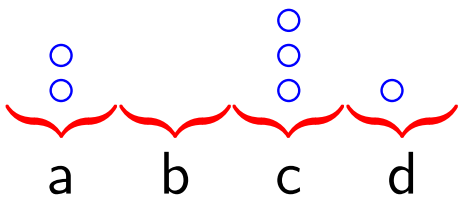

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Answering Q1–Q4

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Correct order:

Q2.	Order 9 baseball players $(9!)$	362,880
Q3.	Pick-6; numbers 1–40 $\binom{40}{6}$	3,838,380
Q4.	12 donuts from 30 $\left(\!\!\left(\binom{30}{12}\right)\!\!\right)$	7,898,654,920
Q1.	8-character passwords (62^8)	218,340,105,584,896

Summary

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repetition allowed		
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