Course Notes

Combinatorics, Fall 2011

Queens College, Math 636

Prof. Christopher Hanusa

On the web: http://people.qc.cuny.edu/faculty
/ christopher.hanusa/courses/636fa11/
The following are books that I recommend to complement this course. I have asked that they be placed *on reserve* in the library.

Benjamin and Quinn. *Proofs that really count.*  
Bóna. *A walk through combinatorics.*  
Brualdi. *Introductory combinatorics.*  
Graham, Knuth, and Patashnik. *Concrete mathematics.*  
Mazur. *Combinatorics: A guided tour*  
van Lint and Wilson. *A course in combinatorics.*
What is combinatorics?

In this class: Learn how to count ...
What is combinatorics?

In this class: Learn how to count ... better.
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**Question:** How many domino tilings are there of an $8 \times 8$ chessboard?
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How many people think there are more than:

10?
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How many people think there are more than: 

100?
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**Question:** How many domino tilings are there of an $8 \times 8$ chessboard?

How many people think there are more than: 

1,000?
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**Question:** How many domino tilings are there of an $8 \times 8$ chessboard?

How many people think there are more than:

10,000?
What is combinatorics?

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*Question*: How many domino tilings are there of an $8 \times 8$ chessboard?

![Domino tilings examples]

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100,000?
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How many people think there are more than:

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How many people think there are more than:

$10,000,000$?
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The TRUE number is:
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The TRUE number is:

$$12,988,816.$$
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We have the answer!
What is combinatorics?

In this class: Learn how to count ... better.

**Question:** How many domino tilings are there of an $8 \times 8$ chessboard?

The TRUE number is:

12,988,816.

We have the answer! What does it mean?
How to determine the “answer”?  
- Convert the chessboard into a combinatorial structure (a graph).
- Represent the graph numerically as a matrix.
- Take the determinant of this matrix.
Domino tilings

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**Question:** How many domino tilings are there of an $m \times n$ board?

**Answer:** If $m$ and $n$ are both even, then we have the formula (!):

$$
\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left( 4 \cos^2 \frac{\pi j}{m+1} + 4 \cos^2 \frac{\pi k}{n+1} \right).
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Combinatorial questions

Given some discrete objects, what properties and structures do they have?

- Can we count the arrangements?
- Can we enumerate the arrangements?
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  - **Prove optimality**.
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(Requires many proofs.) (Uses a different kind of reasoning!)
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All homeworks online; first one due next Wednesday.
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**Q1.** How many 8-character passwords are there using A–Z, a–z, 0–9?
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**Q1.** How many 8-character passwords are there using A–Z, a–z, 0–9?

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Think Write Pair Share: Order these from smallest to largest.
Counting words

*Definition:* A list or word is an ordered sequence of objects.

*Definition:* A \( k \)-list or \( k \)-word is a list of length \( k \).

- A list is always ordered and a set is always unordered.
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- The first two entries can be either \( A \) or \( B \).
- The last entry is either 5 or 6.
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**Answer:** We can solve this using a tree diagram:
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**Alternatively:** Notice two independent choices for each character. Multiply \(2 \cdot 2 \cdot 2 = 8\).
The Product Principle

This illustrates:

**The product principle**: When counting lists \((l_1, l_2, \ldots, l_k)\),

- **IF** there are \(c_1\) choices for entry \(l_1\), each leading to a different list.
- **AND IF** there are \(c_i\) choices for entry \(l_i\), no matter the choices made for \(l_1\) through \(l_{i-1}\), each leading to a different list.
- **THEN** there are \(c_1c_2\cdots c_k\) such lists.
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**Caution:** The product principle seems simple, but we must be careful when we use it.
Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?

Answer: Creating a word of length 8, with ____ choices for each character. Therefore, the number of 8-character passwords is ____.

(=218,340,105,584,896)
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In general, the number of words of length $k$ that can be made from an alphabet of length $n$ and where repetition is allowed is $n^k$
Application: Counting Subsets

Example. How many subsets of a set $S = \{s_1, s_2, \ldots, s_n\}$ are there?
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- $n = 0$: $S = \emptyset$
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  \item \( n = 1 \): \( S = \{s_1\} \leadsto \{\emptyset, \{s_1\}\} \), size 2.
  \item \( n = 2 \): \( S = \{s_1, s_2\} \leadsto \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}\} \), size 4.
\end{itemize}
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- $n = 3$: $S = \{s_1, s_2, s_3\} \leadsto \left\{ \emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}, \{s_3\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\} \right\}$, 8.

It appears that the number of subsets of $S$ is ______. (notation)
Example. How many subsets of a set $S = \{s_1, s_2, \ldots, s_n\}$ are there?

**Strategy:** “Try small problems, see a pattern.”

- $n = 0$: $S = \emptyset \implies \{\emptyset\}$, size 1.
- $n = 1$: $S = \{s_1\} \implies \{\emptyset, \{s_1\}\}$, size 2.
- $n = 2$: $S = \{s_1, s_2\} \implies \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}\}$, size 4.
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This number also counts __________________________.
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It appears that the number of subsets of $S$ is _____. (notation)

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We can label the subsets by whether or not they contain $s_i$.

For example, for $n = 3$, we label the subsets $\left\{ 000, 100, 010, 011, 100, 101, 011, 111 \right\}$.
Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

*Answer:* The number of choices for each lineup spot are:

___  ___  ___  ___  ___  ___  ___  ___  ___  ___  ___.
Permutations

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Answer: The number of choices for each lineup spot are:

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Multiplying gives that the number of lineups is ____ = 362,880.
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Definition: A permutation of an \( n \)-set \( S \) is an (ordered) list of all elements of \( S \). There are \( n! \) such permutations.

“Permutation” always refers to a list without repetition.
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**Definition:** A **\( k \)-permutation** of an \( n \)-set \( S \) is an (ordered) list of \( k \) distinct elements of \( S \).

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Definition: A \( k \)-permutation of an \( n \)-set \( S \) is an (ordered) list of \( k \) distinct elements of \( S \). How many are there?

- “Permutation” always refers to a list without repetition.
Lists WITHOUT repetition

*Question:* How many 8-character passwords are there using $A–Z$, $a–z$, 0–9, containing no repeated character?

**OK:** 2eas3FGS, 10293465  
**Not OK:** 2kdjng2, oOoOoOo0
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$$_______$$

for a total of $(62)_8 = \frac{62!}{54!}$ passwords.
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In general, the number of words of length \(k\) that can be made from an alphabet of length \(n\) and where repetition is NOT allowed is \((n)_k\).
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**Answer:** The number of choices for each character are:  
\[
\begin{array}{ccccccc}
\_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\
\end{array}
\]
for a total of \((62)^8 = \frac{62!}{54!}\) passwords.

In general, the number of words of length \(k\) that can be made from an alphabet of length \(n\) and where repetition is NOT allowed is \((n)_k\).

- That is, the number of \(k\)-permutations of an \(n\)-set is \((n)_k\).
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- That is, the number of \(k\)-permutations of an \(n\)-set is \((n)_k\).
- Special case: For \(n\)-permutations of an \(n\)-set: \(n!\).
Notation

Some quantities appear frequently, so we use shorthand notation:

- \([n] := \{1, 2, \ldots, n\}\)
- \(2^S := \text{set of all subsets of } S\)
- \(n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1\)
- \((n)_k := n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}\)

★ Leave answers to counting questions in terms of these quantities.

★ Do NOT multiply out unless you are comparing values.
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- $2^S := \text{set of all subsets of } S$
- $n! := n \cdot (n - 1) \cdot (n - 2) \cdots 2 \cdot 1$
- $(n)_k := n \cdot (n - 1) \cdot (n - 2) \cdots (n - k + 1) = \frac{n!}{(n - k)!}$
- $\binom{n}{k} := \frac{n!}{k!(n - k)!} = \frac{(n)_k}{k!}$
- $\binom{n}{k} := \binom{k + n - 1}{k}$

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★ **Do NOT** multiply out unless you are comparing values.
My question: In how many ways are there to choose a subset of \( k \) objects out of a set of \( n \) objects?
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Your answer: \( \binom{n}{k} \). “\( n \) choose \( k \)”. 
Counting subsets of a set

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The individual objects we are counting are unordered. They are subsets, not lists.
A formula for \( \binom{n}{k} \)

You may know that \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \). But why?
A formula for $\binom{n}{k}$

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Let’s rearrange:

$$(n)_k = \binom{n}{k} k!$$
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Let's rearrange:

$\binom{n}{k} = \frac{(n)_k}{k!} = \left(\frac{n}{k}\right) k!$

We ask the question:

“In how many ways are there to create a $k$-list of an $n$-set?”

LHS:

RHS:
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We ask the question:

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LHS:

RHS:

Since we counted the same quantity twice, they must be equal!
Counting Multisets

**Definition:** A multiset is a set where repetition is included.
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Think Write Pair Share: Enumerate all multisubsets of \( [3] \).
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Answer:

How would you describe a \( k \)-multisubset of \( [n] \)?
Stars and Bars

*Question:* How many \( k \)-multisets are there from an \( n \)-set?

— *is the same as* —

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\[
\binom{k + n - 1}{k} =: \binom{n}{k}
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Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?
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Answer: \( \binom{30}{12} = 7,898,654,920. \)

Correct order:

Q2. Order 9 baseball players \((9!)\) 362,880
Q3. Pick-6; numbers 1–40 \(\binom{40}{6}\) 3,838,380
Q4. 12 donuts from 30 \(\binom{30}{12}\) 7,898,654,920
Q1. 8-character passwords \((62^8)\) 218,340,105,584,896
## Summary

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<thead>
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