Patterns in Permutations

Definition: We say that an n-permutation $p = p_1 p_2 \cdots p_n$ *contains the pattern* 132 if there exist integers $i < j < k$ such that $p_i < p_k < p_j$.

Example. The permutation $p = 3561247$ contains the pattern 132 because

Example. The permutation $p = 6723415$ does not contain the pattern 132. Why not?

Definition: We say that an n-permutation $p = p_1 p_2 \cdots p_n$ *avoids the pattern* 132 if there do not exist integers $i < j < k$ s.t. $p_i < p_j < p_k$

Definition: Let $p = p_1 p_2 \cdots p_n$ be an n-permutation
and $q = q_1 q_2 \cdots q_k$ be a k-permutation $(k < n)$.
Then *p contains q* if there exist integers $i_i < i_2 < \cdots < i_k$ such that the entries $p_{i_1}, \ldots, p_{i_k}$ are in the same relative order as $q_1, \ldots, q_k$.
If there are no such integers, we say *p avoids q*.

Counting instances of 132-avoidance

Question. In how many ways can we put n children of different heights in a line so that each child can see all shorter children ahead in line?

Translation. How many n-permutations avoid the pattern 132? (Why?)

Solution. Define c_n to be the number of n-perms avoiding 132 and $C(x)$ to be the generating function $C(x) = \sum_{n \geq 0} c_n x^n$.

Game plan:

- Find an expression for c_n in terms of values of c_k for $k < n$.
- Use this expression to determine an equation that $C(x)$ satisfies.
- Solve this equation to find a compact form for $C(x)$.
- Apply the (gen'l) binomial theorem to find the formula for c_n.

Counting instances of 132-avoidance

Question. How many n-permutations avoid the pattern 132?

- Find an expression for c_n in terms of values of c_k for $k < n$.

The middle of the pattern "132" is the "3". To understand n-perms $p_1 \cdots p_n$ avoiding "132", investigate where the largest entry is—"n".

If $p_i = n$, what can we say about (the entries p_1 through p_{i-1}) and (the entries p_{i+1} through p_n)?

- How they interact:

- Individually:

Therefore the c_n satisfy the recurrence $c_n = \sum_{i=1}^{n} c_{i-1} c_{n-i}$, and $c_0 = 1$.

Counting instances of 132-avoidance

Question. How many n-permutations avoid the pattern 132?

- Use this expression to determine an equation that $C(x)$ satisfies.

Multiply each side by x^n and sum over all n:

$$\begin{aligned}
\sum_{n\geq 1} c_n x^n &= \sum_{n\geq 1}\left(\sum_{i=1}^{n} c_{i-1}c_{n-i}\right)x^n \\
&= x\left[\sum_{n\geq 1}\left(\sum_{j=0}^{n-1} c_j c_{(n-1)-j}\right)x^{n-1}\right] && \text{Reindex } i. \\
&= x\left[\sum_{m\geq 0}\left(\sum_{j=0}^{m} c_j c_{m-j}\right)x^{m}\right] && \text{Reindex } n. \\
C(x)-1 &= xC(x)^2 && \text{Notice } C(x)^2.
\end{aligned}$$

Counting instances of 132-avoidance

Question. How many n-permutations avoid the pattern 132?

- Solve this equation to find a compact form for $C(x)$.

$$xC(x)^2 - C(x) + 1 = 0 \Longrightarrow C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Uh-oh! Two solutions? Which one to pick? We know $C(0) = c_0 = 1$, so plug in $x = 0$ and determine

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

- Apply the (gen'l) binomial theorem to find the formula for c_n.

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k},$$

for $\alpha \in \mathbb{R}$ and $\binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k(k-1)\cdots(1)}$,

Counting instances of 132-avoidance

Question. How many n-permutations avoid the pattern 132?

- Apply the (gen'l) binomial theorem to find the formula for c_n:

$$C(x) = (1 - \sqrt{1-4x})/2x.$$

$$\begin{aligned}
\sqrt{1-4x} &= \left((-4x)+1\right)^{1/2} = \textstyle\sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k && \text{Apply bin. thm.}\\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-4x)^k && \text{Write out } \binom{\alpha}{k}.\\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!}(-1)^k 4^k x^k && \text{Expand } \binom{\alpha}{k}.\\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k}(-1)^k 4^k x^k && \text{Combine powers of 2.}\\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-3)\cdot(2k-2)}{k!\cdot 2\cdot 4\cdots(2k-2)} 2^k x^k && \text{Insert \#s to make factorials.}\\
&= 1 + \textstyle\sum_{k=1}^{\infty} -\frac{(2k-2)!}{k!(2^{k-1})1\cdot 2\cdots(k-1)} 2^k x^k && \text{Write factorials, more 2 powers.}\\
&= 1 + \textstyle\sum_{k=1}^{\infty} \frac{-2}{k}\frac{(2k-2)!}{(k-1)!(k-1)!} x^k && \text{Simplify.}
\end{aligned}$$

$C(x) = \sum_{k=1}^{\infty} \frac{1}{k}\binom{2(k-1)}{k-1}x^{k-1} = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^n$, and $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Avoiding other patterns

Notation: Define $S_n(q)$ to be the number of q-avoiding n-perms.

We determined that $S_n(132) = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number.

What about other patterns? 123, 132, 213, 231, 312, 321.

Claim 1. $S_n(231) = S_n(132)$.

Definition: For a permutation $p = p_1 p_2 \cdots p_n$, the *reverse* of p is the permutation $p' = p_n \cdots p_2 p_1$.
Example. The reverse of the permutation $q = 132$ is $q' = 231$.

If the n-permutation p avoids 132, then the reverse of p avoids 231. This gives a bijection between n-permutations avoiding 132 and those avoiding 231.

In general, if q' is the reverse of q, then $S_n(q') = S_n(q)$.
We conlude $S_n(123) = S_n(321)$ and $S_n(312) = S_n(213)$.

Avoiding other patterns

Pattern list: (123 and 321), (132 and 231), (213 and 312).

Claim 2. $S_n(312) = S_n(132)$.

Definition: For a permutation $p = p_1 p_2 \cdots p_n$, the *complement* of p is the permutation $\bar{p} = (n+1-p_1)(n+1-p_2)\cdots(n+1-p_n)$.
Example. $\bar{q} = 312$ is the complement of $q = 132$.
Example. $\bar{r} = 31254$ is the complement of $r = 35412$.

If the n-permutation p avoids 132, then the complement of p avoids 312. This gives a bijection between n-permutations avoiding 132 and those avoiding 312.

In general, if $\bar{q}$ is the complement of q, then $S_n(\bar{q}) = S_n(q)$.

Avoiding other patterns

Pattern list: [(123 and 321)], [(132 and 231) and (213 and 312)]

Claim 3. $S_n(123) = S_n(132)$. (Now we have to do some work!)

Goal: Bijection between perms avoiding 123 and perms avoiding 132.

Definition: In a permutation $p = p_1 p_2 \cdots p_n$, a *left-to-right minimum* is an entry p_i which is smaller than all entries before it.

Example. The left-to-right minima of 68357412 are 6, 3, and 1.

Key Observations:

- The left-to-right minima form a decreasing subsequence.
- (The union of two decreasing subsequences) avoids 123.

What does a 123-avoiding perm look like?	What does a 132-avoiding perm look like?
Remove l-t-r min; what's left? Remaining entries decreasing.	Remove l-t-r min; what's left? All entries after a l-t-r minimum must be increasing.

A function from 123-avoiding perms to 132-avoiding perms

We will create a bijection f between 123-av perms and 132-av perms.

Let p be a 123-avoiding permutation (entries other than l-t-r minima decreasing). Define $f(p)$ by fixing the left-to-right minima and reinserting entries from left to right: at each step insert the smallest remaining entry which is larger than the previous minimum.

Example. $f(68472531) =$

Claim: $f(p)$ is 132-avoiding.

A function from 132-avoiding perms to 123-avoiding perms

Let q be a 132-avoiding permutation. Define $g(q)$ by fixing the left-to-right minima and reinserting the remaining entries from left to right in decreasing order. Therefore $g(q)$ is a 123-avoiding permutation and $g(f(p)) = p$.

Example. $f(67452381) =$

Conclusion. We have a bijection and therefore, $S_n(123) = S_n(132)$.

Conclusion. For all patterns q of length 3, $S_n(q) = c_n$.

Remark. This is not the case for patterns of length four.

- $S_n(1324) : 1, 2, 6, 23, 103, 512, 2740, 15485$
- $S_n(1234) : 1, 2, 6, 23, 103, 513, 2761, 15767$
- $S_n(1324) : 1, 2, 6, 23, 103, 513, 2762, 15793$