Patterns in Permutations

**Definition:** We say that an \( n \)-permutation \( p = p_1p_2 \cdots p_n \) contains the pattern 132 if there exist integers \( i < j < k \) such that \( p_i < p_k < p_j \).

**Example.** The permutation \( p = 3561247 \) contains the pattern 132 because

**Example.** The permutation \( p = 6723415 \) does not contain the pattern 132. Why not?

**Definition:** We say that an \( n \)-permutation \( p = p_1p_2 \cdots p_n \) avoids the pattern 132 if there do not exist integers \( i < j < k \) s.t. \( p_i < p_j < p_k \).

**Definition:** Let \( p = p_1p_2 \cdots p_n \) be an \( n \)-permutation and \( q = q_1q_2 \cdots q_k \) be a \( k \)-permutation \( (k < n) \). Then \( p \) contains \( q \) if there exist integers \( i_1 < i_2 < \cdots < i_k \) such that the entries \( p_{i_1}, \ldots, p_{i_k} \) are in the same relative order as \( q_1, \ldots, q_k \). If there are no such integers, we say \( p \) avoids \( q \).
Counting instances of 132-avoidance

Question. In how many ways can we put \( n \) children of different heights in a line so that each child can see all shorter children ahead in line?

Translation. How many \( n \)-permutations avoid the pattern 132? (Why?)

Solution. Define \( c_n \) to be the number of \( n \)-perms avoiding 132 and \( C(x) \) to be the generating function \( C(x) = \sum_{n \geq 0} c_n x^n \).

Game plan:

- Find an expression for \( c_n \) in terms of values of \( c_k \) for \( k < n \).
- Use this expression to determine an equation that \( C(x) \) satisfies.
- Solve this equation to find a compact form for \( C(x) \).
- Apply the (gen’l) binomial theorem to find the formula for \( c_n \).
Counting instances of 132-avoidance

**Question.** How many $n$-permutations avoid the pattern 132?

- Find an expression for $c_n$ in terms of values of $c_k$ for $k < n$.

The middle of the pattern “132” is the “3”. To understand $n$-perms $p_1 \cdots p_n$ avoiding “132”, investigate where the largest entry is—“n”.

If $p_i = n$, what can we say about (the entries $p_1$ through $p_{i-1}$) and (the entries $p_{i+1}$ through $p_n$)?

- How they interact:

- Individually:

Therefore the $c_n$ satisfy the recurrence $c_n = \sum_{i=1}^{n} c_{i-1} c_{n-i}$, and $c_0 = 1$. 
Counting instances of 132-avoidance

Question. How many $n$-permutations avoid the pattern 132?

Use this expression to determine an equation that $C(x)$ satisfies.

Multiply each side by $x^n$ and sum over all $n$:

$$\sum_{n \geq 1} c_n x^n = \sum_{n \geq 1} \left( \sum_{i=1}^{n} c_{i-1} c_{n-i} \right) x^n$$

$$= x \left[ \sum_{n \geq 1} \left( \sum_{j=0}^{n-1} c_j c_{(n-1)-j} \right) x^{n-1} \right]$$

Reindex $i$.

$$= x \left[ \sum_{m \geq 0} \left( \sum_{j=0}^{m} c_j c_{m-j} \right) x^m \right]$$

Reindex $n$.

$$C(x) - 1 = xC(x)^2$$

Notice $C(x)^2$. 
Counting instances of 132-avoidance

**Question.** How many $n$-permutations avoid the pattern 132?

- Solve this equation to find a compact form for $C(x)$.

$$xC(x)^2 - C(x) + 1 = 0 \implies C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$  

Uh-oh! Two solutions? Which one to pick? We know $C(0) = c_0 = 1$, so plug in $x = 0$ and determine

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$  

- Apply the (gen’l) binomial theorem to find the formula for $c_n$.

$$(x + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha - k},$$  

for $\alpha \in \mathbb{R}$ and \( \binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k(k - 1) \cdots (1)} \),
Counting instances of 132-avoidance

**Question.** How many $n$-permutations avoid the pattern 132?

- Apply the (gen’l) binomial theorem to find the formula for $c_n$:

$$C(x) = \frac{(1 - \sqrt{1 - 4x})}{2x}.$$ 

\[
\begin{align*}
\sqrt{1 - 4x} &= ((-4x) + 1)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k \\
&= 1 + \sum_{k=1}^{\infty} \binom{1/2}{k} \frac{\left(\frac{1}{2} - 1\right) \cdots \left(\frac{1}{2} - k + 1\right)}{k!} (-4x)^k \\
&= 1 + \sum_{k=1}^{\infty} \binom{1}{k} \frac{\left(-\frac{1}{2}\right) \cdots \left(-\frac{2k-3}{2}\right)}{k!} (-1)^k 4^k x^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{(1)(2k-3) \cdots 1}{2^k} (-1)^k 4^k x^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-3)(2k-2)}{k!} \frac{2^k}{k!} x^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-3)(2k-2)}{k!(2k-2)!} x^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{2}{(k-1)!(k-1)!} x^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{k} \frac{2}{k} \left(\frac{2k-2}{(k-1)!(k-1)!}\right) x^k.
\end{align*}
\]

Therefore, $C(x) = \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} x^{k-1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n$, and $c_n = \frac{1}{n+1} \binom{2n}{n}$. 
Avoiding other patterns

**Notation:** Define $S_n(q)$ to be the number of $q$-avoiding $n$-perms.

We determined that $S_n(132) = \frac{1}{n+1} \binom{2n}{n}$, the $n$th Catalan number.

What about other patterns? 123, 132, 213, 231, 312, 321.

**Claim 1.** $S_n(231) = S_n(132)$.

**Definition:** For a permutation $p = p_1 p_2 \cdots p_n$, the *reverse* of $p$ is the permutation $p' = p_n \cdots p_2 p_1$.

**Example.** The reverse of the permutation $q = 132$ is $q' = 231$.

If the $n$-permutation $p$ avoids 132, then the reverse of $p$ avoids 231. This gives a bijection between $n$-permutations avoiding 132 and those avoiding 231.

In general, if $q'$ is the reverse of $q$, then $S_n(q') = S_n(q)$.

We conclude $S_n(123) = S_n(321)$ and $S_n(312) = S_n(213)$. 


Avoiding other patterns

Pattern list: (123 and 321), (132 and 231), (213 and 312).

Claim 2. $S_n(312) = S_n(132)$.

Definition: For a permutation $p = p_1 p_2 \cdots p_n$, the complement of $p$ is the permutation $\bar{p} = (n+1-p_1)(n+1-p_2)\cdots(n+1-p_n)$.

Example. $\bar{q} = 312$ is the complement of $q = 132$.

Example. $\bar{r} = 31254$ is the complement of $r = 35412$.

If the $n$-permutation $p$ avoids 132, then the complement of $p$ avoids 312. This gives a bijection between $n$-permutations avoiding 132 and those avoiding 312.

In general, if $\bar{q}$ is the complement of $q$, then $S_n(\bar{q}) = S_n(q)$. 
Avoiding other patterns

Pattern list: [(123 and 321)], [(132 and 231) and (213 and 312)]

Claim 3. $S_n(123) = S_n(132)$.  
(Now we have to do some work!)

Goal: Bijection between perms avoiding 123 and perms avoiding 132.

Definition: In a permutation $p = p_1p_2\cdots p_n$, a left-to-right minimum is an entry $p_i$ which is smaller than all entries before it.

Example. The left-to-right minima of 68357412 are 6, 3, and 1.

Key Observations:
- The left-to-right minima form a decreasing subsequence.
- (The union of two decreasing subsequences) avoids 123.

<table>
<thead>
<tr>
<th>What does a 123-avoiding perm look like?</th>
<th>What does a 132-avoiding perm look like?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Remove l-t-r min; what’s left?</td>
<td>Remove l-t-r min; what’s left?</td>
</tr>
<tr>
<td>Remaining entries decreasing.</td>
<td>All entries after a l-t-r minimum must be increasing.</td>
</tr>
</tbody>
</table>
A function from 123-avoiding perms to 132-avoiding perms

We will create a bijection $f$ between 123-av perms and 132-av perms.

Let $p$ be a 123-avoiding permutation (entries other than l-t-r minima decreasing). Define $f(p)$ by fixing the left-to-right minima and reinserting entries from left to right: at each step insert the smallest remaining entry which is larger than the previous minimum.

Example. $f(68472531) =$

Claim: $f(p)$ is 132-avoiding.
A function from 132-avoiding perms to 123-avoiding perms

Let \( q \) be a 132-avoiding permutation. Define \( g(q) \) by fixing the left-to-right minima and reinserting the remaining entries from left to right in decreasing order. Therefore \( g(q) \) is a 123-avoiding permutation and \( g(f(p)) = p \).

**Example.** \( f(67452381) = \)

**Conclusion.** We have a bijection and therefore, \( S_n(123) = S_n(132) \).

**Conclusion.** For all patterns \( q \) of length 3, \( S_n(q) = c_n \).

**Remark.** This is not the case for patterns of length four.

- \( S_n(1324) : 1, 2, 6, 23, 103, 512, 2740, 15485 \)
- \( S_n(1234) : 1, 2, 6, 23, 103, 513, 2761, 15767 \)
- \( S_n(1324) : 1, 2, 6, 23, 103, 513, 2762, 15793 \)