Knight’s Tours

In chess, a knight (\(\text{♞} \)) is a piece that moves in an “L”: two spaces over and one space to the side.

Question Is it possible for a knight to start on some square and, by a series of valid knight moves, visit each square on an \(8 \times 8\) chessboard once? (How about return to where it started?)

Definition: A path of the first kind is called an open knight’s tour. A cycle of the second kind is called a closed knight’s tour.
Question: Are there any knight's tours on an $m \times n$ chessboard?
The Graph Theory of Knight’s Tours

For any board we can draw a corresponding knight move graph: Create a vertex for every square on the board and create edges between vertices that are a knight’s move away.

An open/closed knight’s tour
on the board

A knight move always alternates between white and black squares. Therefore, a knight move graph is always ________________.

Question Are there any knight’s tours on an $m \times n$ chessboard?
Knight’s Tour Theorem

Theorem An $m \times n$ chessboard with $m \leq n$ has a closed knight’s tour unless one or more of these conditions holds:

1. $m$ and $n$ are both odd.
2. $m = 1$, 2, or 4.
3. $m = 3$ and $n = 4$, 6, or 8.

“Proof” We will only show that it is impossible in these cases.

Case 1. When $m$ and $n$ are both odd,

Case 2. When $m = 1$ or 2, the knight move graph is not connected.
Knight’s Tour Theorem

**Case 2.** When \( m = 4 \), draw the knight move graph \( G \).

Suppose there exists a Hamiltonian cycle \( C \) in the graph \( G \). Since \( G \) is bipartite, \( C \) alternates between white and black vertices.

Notice that every red vertex in \( C \) is adjacent to only blue vertices. And, there are the same number of red and blue vertices.

So, \( C \) must alternate between red and blue vertices. This means: All vertices of \( C \) are “white and red” or “black and blue”.
**Knight’s Tour Theorem**

**Case 3.** $3 \times 4$ is covered by Case 2. Consider the $3 \times 6$ board:

Assume that there is a Hamiltonian cycle $C$ in $G$. Then, $C$ visits each vertex $v$ and uses two of $v$’s incident edges. If $\deg(v) = 2$, then both of $v$’s incident edges are in $C$. Draw in all these “forced edges” above. With just these forced edges, there is already a cycle $C'$ of length four. This cycle $C'$ cannot be a subgraph of any Hamiltonian cycle, contradicting its existence.

The $3 \times 8$ case is similar, and part of your homework.

See also: “Knight’s Tours on a Torus”, by J. J. Watkins, R. L. Hoenigman