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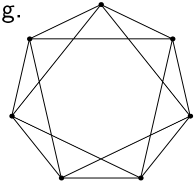

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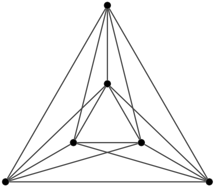

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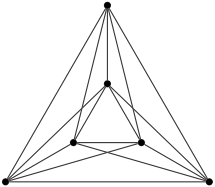

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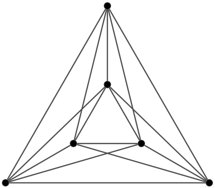

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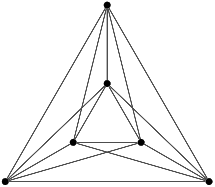

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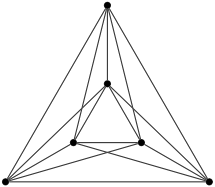

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- This would give a plane drawing of K_5, a contradiction!

Therefore, $cr(K_6) = 3$.

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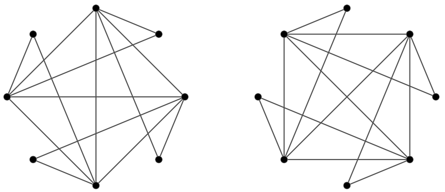

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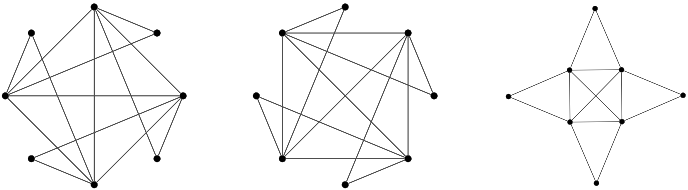

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Fact: $\theta(K_n) = \begin{cases} \left\lfloor \frac{n+7}{6} \right\rfloor & n \neq 9, 10 \\ 3 & n = 9, 10 \end{cases}$ Proved by Beineke, Harary, Vasak, Alekseev, Gonchakov

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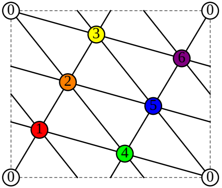

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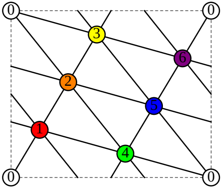

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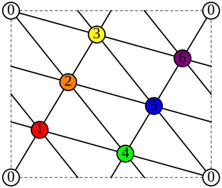

However, we can't embed K_8 on a torus. Perhaps on a surface of genus g?

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Embedding on higher genus surfaces changes Euler's formula!

Theorem. Let G be a graph of genus g. Suppose you have an embedding of G on a surface of genus g with no crossings.
If r is the number of regions, then $p - q + r = \mathbf{2 - 2g}$.

Example. In our embedding of K_5 on the torus (genus 1):

Complete graphs

Planarity statistics for complete graphs:

Statistic	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\text{cr}(K_n)$	0	1	3	9	18	36	60	100	150	225	?	?	?	?	?
$\theta(K_n)$	1	2	2	2	2	**3**	**3**	3	3	3	3	3	3	4	4
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The crossing number of a complete graph is unknown for $n \geq 13$.

Conjecture. (Guy, 1972) The crossing number of a complete graph is

$$cr(G) = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

The cases $cr(K_{11}) = 100$ and $cr(K_{12}) = 150$ were proved in **2007**.