The Six Color Theorem

Theorem. Let G be a planar graph.
There exists a proper 6-coloring of G.

Proof. Let G be a the smallest planar graph (by number of vertices) that has no proper 6-coloring.

By Theorem 8.1.7, there exists a vertex v in G that has degree five or less. $G \setminus v$ is a planar graph smaller than G, so it has a proper 6-coloring.

Color the vertices of $G \setminus v$ with six colors; the neighbors of v in G are colored by at most five different colors.

We can color v with a color not used to color the neighbors of v, and we have a proper 6-coloring of G, contradicting the definition of G.

The Five Color Theorem

Theorem. Let G be a planar graph.
There exists a proper 5-coloring of G.

Proof. Let G be a the smallest planar graph (by number of vertices) that has no proper 5-coloring.

By Theorem 8.1.7, there exists a vertex v in G that has degree five or less. $G \setminus v$ is a planar graph smaller than G, so it has a proper 5-coloring.

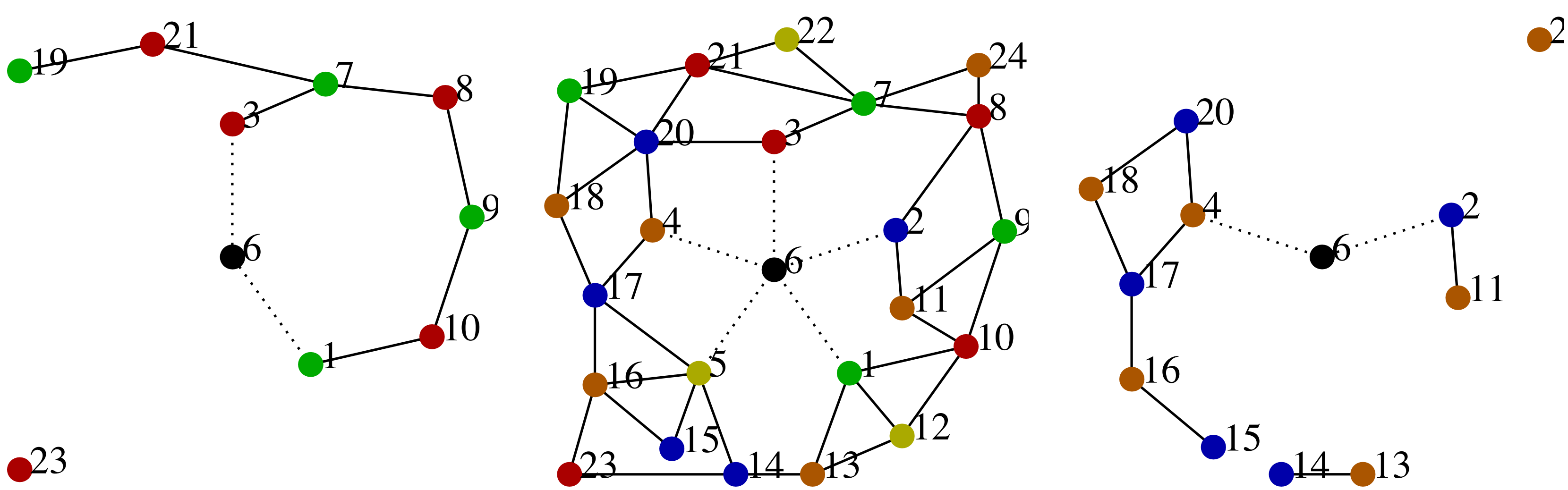
Color the vertices of $G \setminus v$ with five colors; the neighbors of v in G are colored by at most five different colors.

If they are colored with only four colors,
we can color v with a color not used to color the neighbors of v, and we have a proper 5-coloring of G, contradicting the definition of G.

The Kempe Chains Argument

Otherwise the neighbors of v are all colored differently. We will work to modify the coloring on $G \setminus v$ so that only four colors are used.

Consider the subgraphs $H_{1,3}$ and $H_{2,4}$ of $G \setminus v$ constructed as follows:
Let $V_{1,3}$ be the set of vertices in $G \setminus v$ colored with colors 1 or 3.
Let $V_{2,4}$ be the set of vertices in $G \setminus v$ colored with colors 2 or 4.
Let $H_{1,3}$ be the induced subgraph of G on $V_{1,3}$. (Define $H_{2,4}$ similarly)

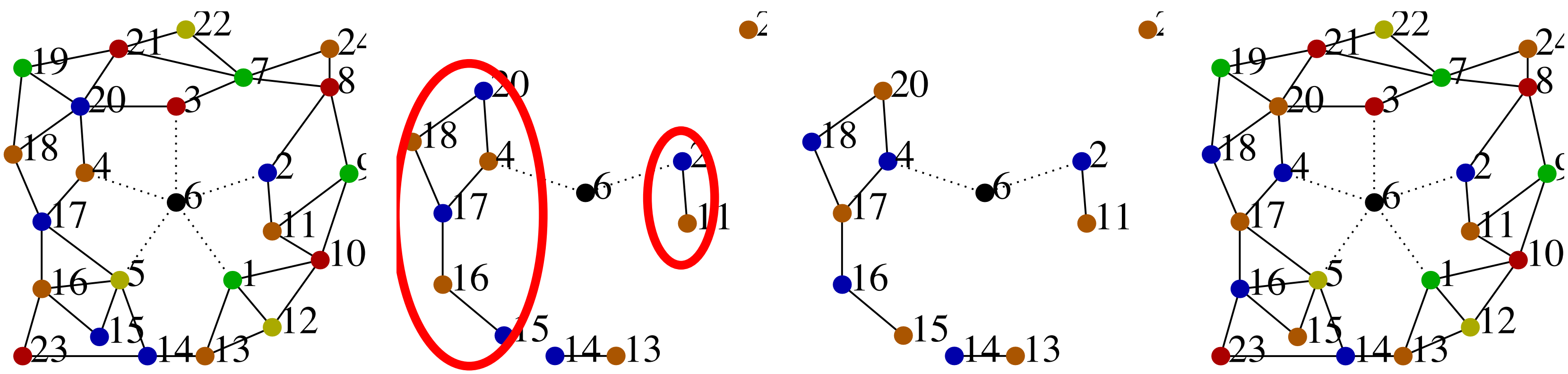


The Kempe Chains Argument

Definition: A **Kempe chain** is a path in $G \setminus v$ between two non-consecutive neighbors of v such that the colors on the vertices of the path *alternate* between the colors on those two neighbors.

In the example above, $3 \to 7 \to 8 \to 9 \to 10 \to 1$ is a Kempe chain: the colors alternate between red and green and 1&3 not consecutive.

Either v_1 and v_3 are in the same component of $H_{1,3}$ or not.
If they are, there is a Kempe chain between v_1 and v_3.
If they are not, (say v_1 is in component C_1 and v_3 is in C_3) then swap colors 1 and 3 in C_1. (Here we show C_2 and C_4)



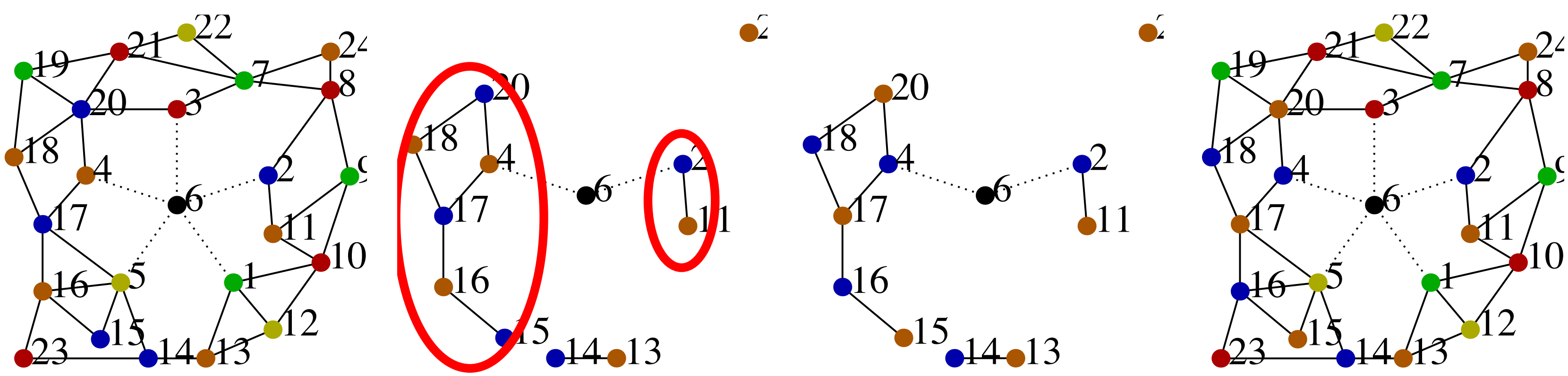
The Kempe Chains Argument

Claim. This remains a proper coloring of $G \setminus v$.

Proof. We need to check that the recoloring does not have two like-colored vertices adjacent.

In C_1, there are only vertices of color 1 and 3 and recoloring does not change that no two adjacent vertices are colored differently.

And, by construction, no vertex adjacent to a vertex in C_1 is colored 1 or 3. This is true before AND after recoloring. □

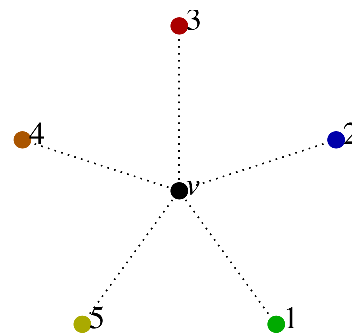


The Kempe Chains Argument

So **either** there is a Kempe chain between v_1 and v_3 **or** we can swap colors so that v's neighbors are colored only using four colors.

Similarly, **either** there is a Kempe chain between v_2 and v_4 **or** we can swap colors to color v's neighbors with only four colors.

Question. Can we have both a v_1-v_3 and a v_2-v_4 Kempe chain?



There are no edge crossings in the graph drawing, so there would exist a vertex____.
This can not exist, so it must be possible to swap colors and be able to place a fifth color on v, contradicting the definition of G.

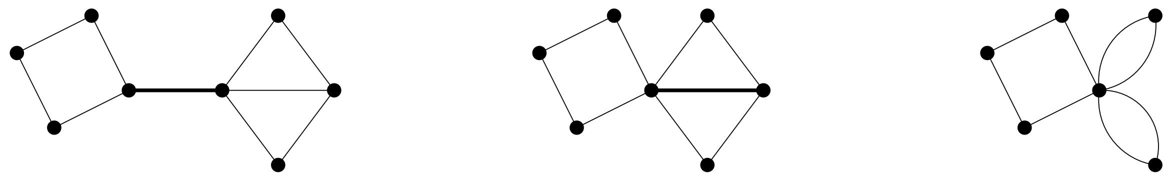
Modifications of Graphs

Definition: **Deletion**

$G \setminus v$ (G delete v): Remove v from the graph and all incident edges.
$G \setminus e$ (G delete e): Remove e from the graph.

Definition: **Contraction**

G/e (G contract e): If $e = vw$, coalesce v and w into a super-vertex adjacent to all neighbors of v and w. [*This may produce a multigraph.*]



Definition: A graph H is a **minor** of a graph G if H can be obtained from G by a sequence of edge deletions and/or edge contractions. [*"Minor" suggests smaller: H is smaller than G.*]

Note. Any subgraph of G is also a minor of G.

Modifications of Graphs

Definition: A **subdivision** of an edge e is the replacement of e by a path of length *at least* two. [*Like adding vertices in the middle of* e.]

Definition: A **subdivision** of a graph H is the result of zero or more sequential subdivisions of edges of H.

Note. If G is a subdivision of H, then G is at least as large as H.

Note. If G is a subdivision of H, then H is a minor of G.
(Contract any edges that had been subdivided!)

Note. The converse is not necessarily true.

Kuratowski's Theorem

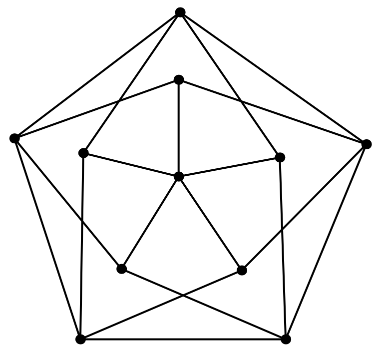
Theorem. Let H be a subgraph of G. If H is nonplanar, then G is nonplanar.

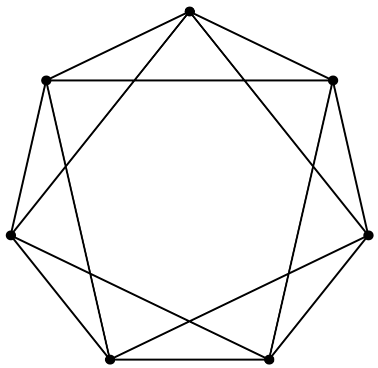
Theorem. Let G be a subdivision of H. If H is nonplanar, then G is nonplanar.

Corollary. If G contains a subdivision of a nonplanar graph, then G is nonplanar.

Theorem. (Kuratowski, 1930) A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Theorem. (Kuratowski variant) A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.





Kuratowski's Theorem

- To prove that a graph G is planar, find a planar embedding of G.
- To prove that a graph G is non-planar, **(a)** Use $q \leq 3p - 6$, or **(b)** find a subgraph of G that is isomorphic to a subdivision of K_5 or $K_{3,3}$, or **(b)** successively delete and contract edges of G to show that K_5 or $K_{3,3}$ is a minor of G.
- Practice on the Petersen graph. (Here, have some copies!)