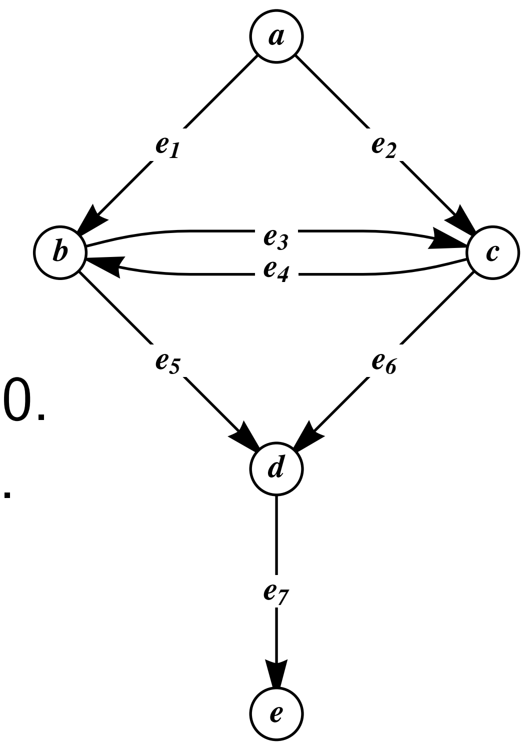
Directed Graphs

Definition: A **directed graph** (or **digraph**) is a graph $G = (V, E)$, where each edge $e = vw$ is directed from one vertex to another:

$$e : v \to w \qquad \text{or} \qquad e : w \to v.$$

Remark. The edge $e : v \to w$ is different from $e' : w \to v$ and a digraph including both is not considered to have multiple edges.

Definition: The **in-degree** of a vertex v is the number of edges directed *toward* v.
Definition: The **out-degree** of a vertex v is the number of edges directed *away from* v.



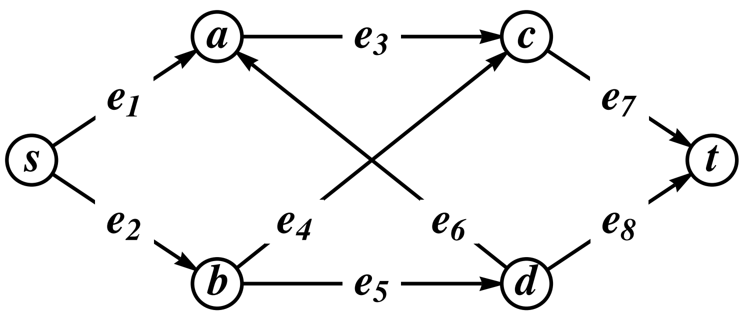
Definition: A **source** s is a vertex with in-degree 0.
Definition: A **sink** t is a vertex with out-degree 0.

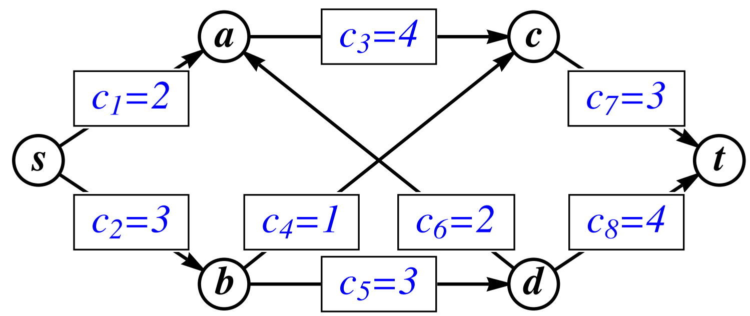
Important. Any **path** or **cycle** in a digraph must respect the direction on each edge.

Network Flows

Definition: A **network** is a directed graph with additional structure:

- There are two distinguished vertices, s (a source) and t (a sink).
- Each edge e has a **capacity** c_e. [*Some sort of limit on flow.*]





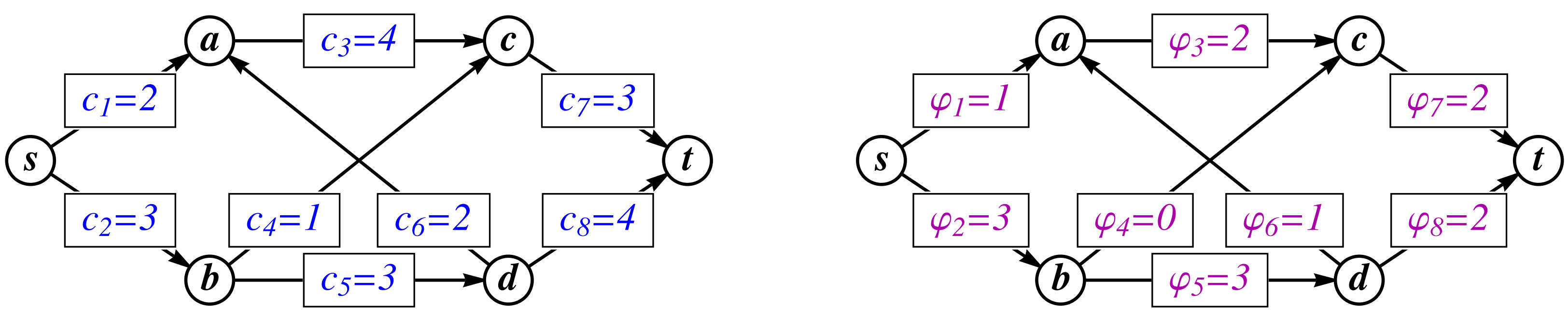
Idea. Graph networks represent real-world networks such as traffic, water, communication, etc.

Goal: Send as much "stuff" from s to t while respecting capacities.

Network Flows

Definition: Given a network G, a **flow** $\vec{\varphi} = \{\varphi_e\}_{e \in E(G)}$ on G is an assignment of values φ_e to every edge of G satisfying:

- $\boxed{0 \le \varphi_e \le c_e}$ for every edge $e \in E(G)$.
  - *The flow respects the capacities.*
- $\boxed{\sum_{e \text{ into } v} \varphi_e = \sum_{e \text{ out of } v} \varphi_e}$ for every vertex $v \in V(G)$ except s or t.
  - *Obeys "conservation of flow" except at s and t.*



Definition: When $\varphi_e = c_e$, we say that e is **saturated**, or **at capacity**.

Maximum Flow

Theorem. Given a flow $\vec{\varphi}$ on a network G, the net flow out of s is equal to the net flow into t. Symbolically, $\sum_{e \text{ out of } s} \varphi_e = \sum_{e \text{ into } t} \varphi_e$.

Proof. Create a new network G' by adding to G an edge $e_\infty : t \to s$ with infinite capacity, and place flow

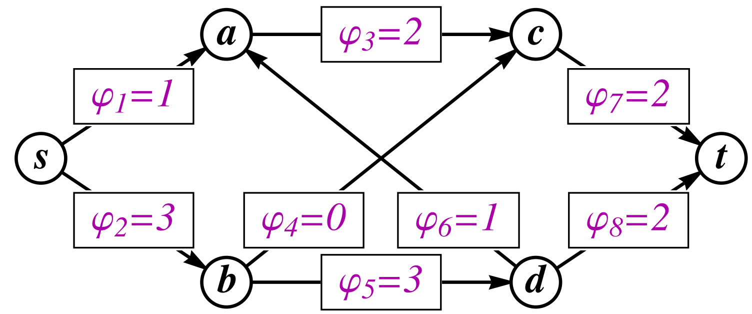
$$\varphi_\infty = \sum_{e \text{ out of } s} \varphi_e$$

on e_∞.

In G', flow is now conserved at every vertex except possibly t. By Kirchhoff's Global Current Law (Theorem 6.2.2), flow must be conserved at t as well.

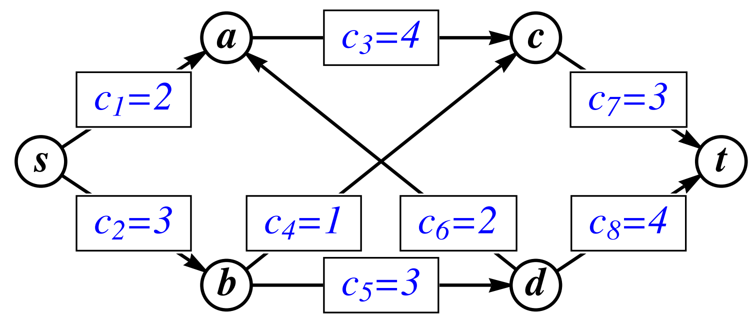
Maximum Flow

Definition: The **throughput** or **value** of a flow $\vec{\varphi}$ is $\sum_{e \text{ out of } s} \varphi_e$, denoted $|\vec{\varphi}|$.



Idea: The throughput is the amount of "stuff" flowing through G.

In our example, $|\vec{\varphi}| =$ ________________.



Goal: For a given network, find the flow with the largest throughput.

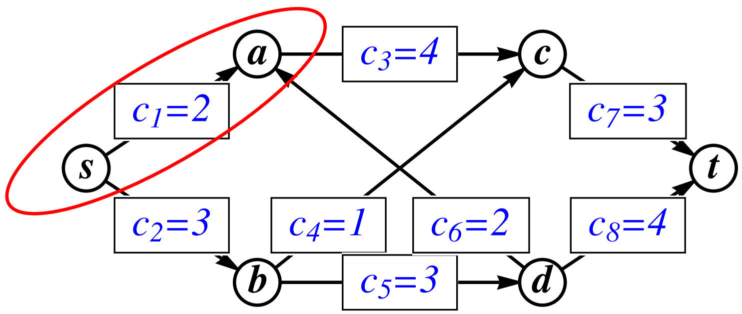
This problem is called **maximum flow**.

MAX FLOW $\quad \underset{\text{over all flows } \vec{\varphi} \text{ on } G}{\text{maximize}} \quad |\vec{\varphi}|$

st-Cuts

A related problem in network theory has to do with st-cuts.

Definition: Let G be a network. Let X be a set of vertices containing s and not containing t. An st-**cut** $[X, X^c]$ is the **set of edges** between a vertex in X and a vertex in X^c (in either direction).



$X =$

$X^c =$

$[X, X^c] =$

$\big|[X, X^c]\big| =$

Definition: The **capacity** of an st-cut, denoted $\big|[X, X^c]\big|$ is the sum of the capacities of the edges **from** a vertex in X **to** a vertex in X^c.

Idea: The capacity of a cut is a limit for how much "stuff" can go from X to X^c.

⋆ Do **not** subtract the capacities of the edges going the other way. ⋆

Max Flow / Min Cut

Goal: For a given network, find the st-cut with the smallest capacity.

This problem is called **minimum cut**.

$$\text{MIN CUT} \qquad \underset{\text{over all cuts } [X, X^c] \text{ on } G}{\text{minimize}} \quad |[X, X^c]|$$

The problems Max Flow and Min Cut are related because for any flow $\vec{\varphi}$, the net flow through the edges of any st-cut $[X, X^c]$ is at most the capacity of $[X, X^c]$. This proves:

Theorem. For any flow $\vec{\varphi}$ and st-cut $[X, X^c]$, $|\vec{\varphi}| \leq |[X, X^c]|$.

Theorem. For any maximum flow $\vec{\varphi}^*$ and minimum st-cut $[X^*, X^{*c}]$,

$$|\vec{\varphi}^*| \leq |[X^*, X^{*c}]|.$$

So, if there exists a flow $\vec{\varphi}$ and st-cut $[X^*, X^{*c}]$ where equality holds, then $\vec{\varphi}$ is a maximum flow and $[X^*, X^{*c}]$ is a minimum cut

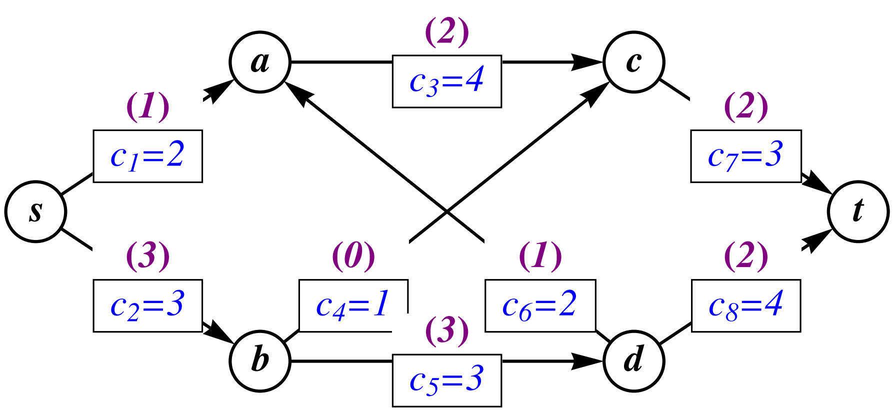
Max Flow / Min Cut Theorem

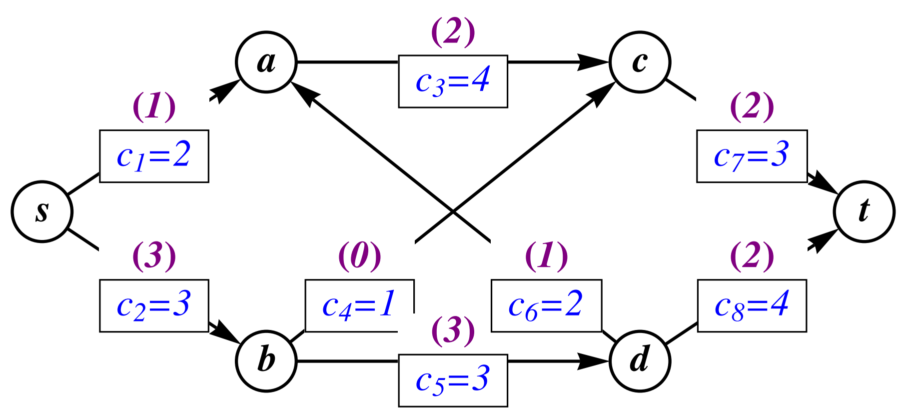
Theorem. (Ford, Fulkerson, 1955) In any network G, the value of any maximum flow is equal to the capacity of any minimum cut.

Proof. Use the Ford–Fulkerson Algorithm to find a max flow.

Idea: Similar to the Hungarian Algorithm for finding a max matching, we will augment an existing flow $\vec{\varphi}$.

Question. What does it look like to *augment a flow*?





We can augment in the forward direction when ____________________.
We can augment in the backward direction when ____________________.

We'll create a *companion graph* to keep track of augmenting paths.

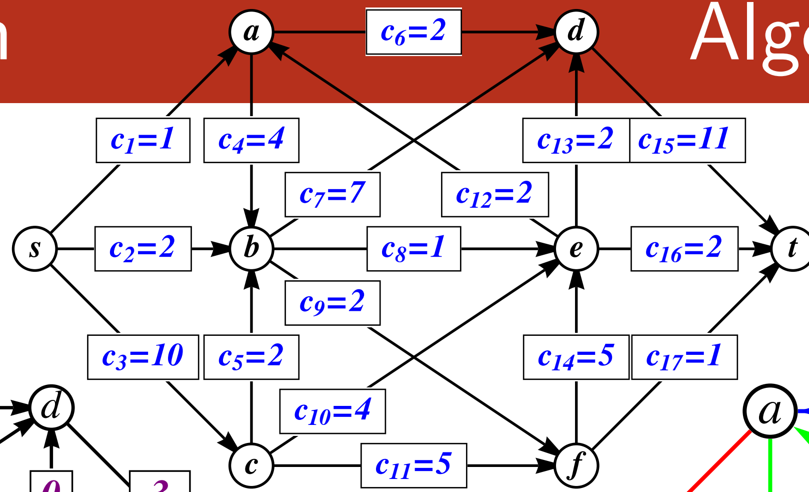
Max Flow / Min Cut Theorem

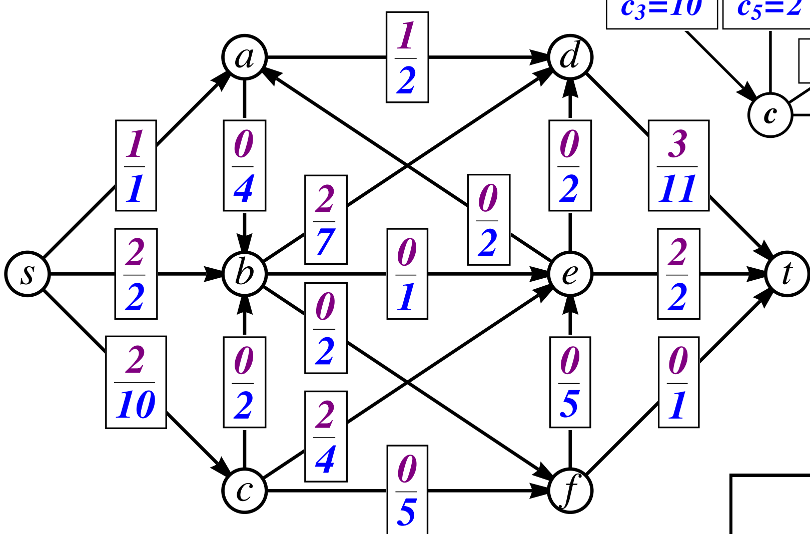
Theorem. (Ford, Fulkerson, 1955) In any network G, the value of any maximum flow is equal to the capacity of any minimum cut.

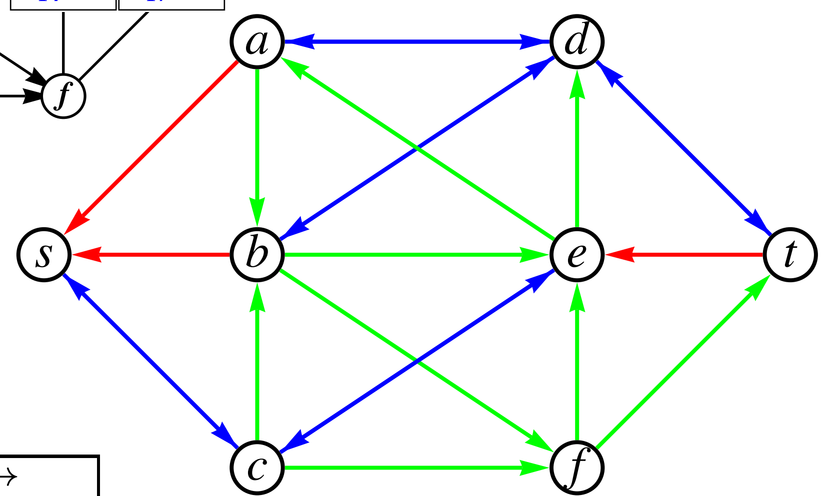
Proof. Use the **Ford–Fulkerson Algorithm**, which finds a max flow.

1. Start with any flow $\vec{\varphi}$ on G.
2. Draw the **flow companion graph** using the underlying graph
   - If $\varphi_e = 0$, orient the edge e forward only.
   - If $0 < \varphi_e < c_e$, orient the edge e both forward and backward.
   - $\varphi_e = c_e$, orient the edge e backward only.
3. $\star$ If there is an st-path in the flow companion graph, send as many units of flow as possible through this path. Repeat Step 2.
   $\star$ If there is no st-path in the flow companion graph, STOP.
   $\rightarrow$ Upon STOP, the current flow is a maximum flow. $\leftarrow$
   In addition, let X be the set of vertices reachable from s in the flow companion graph. Then $[X, X^c]$ is a minimum st-cut.

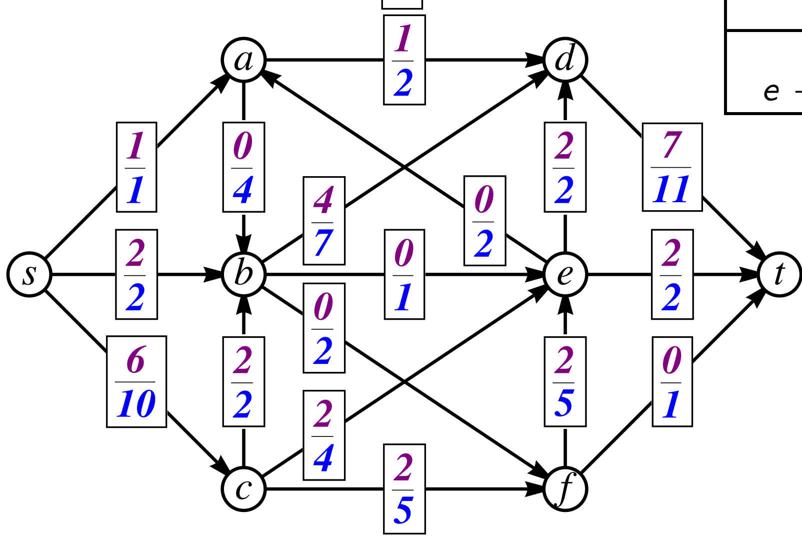
A Ford–Fulkerson Algorithm Example

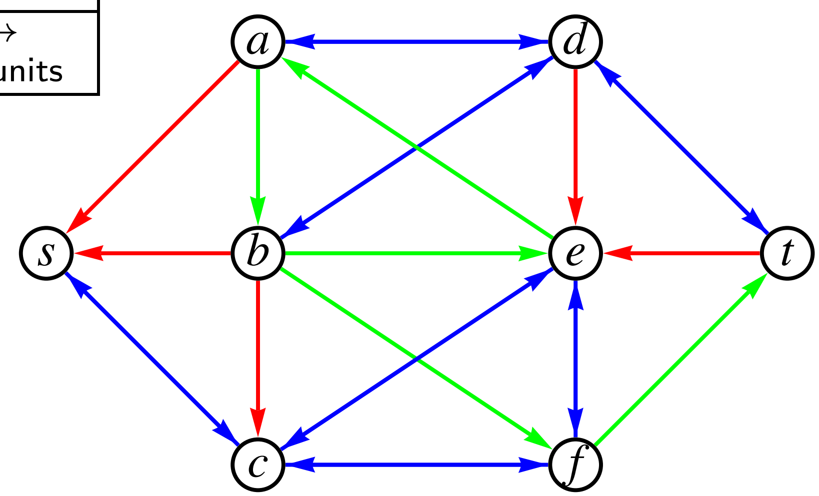




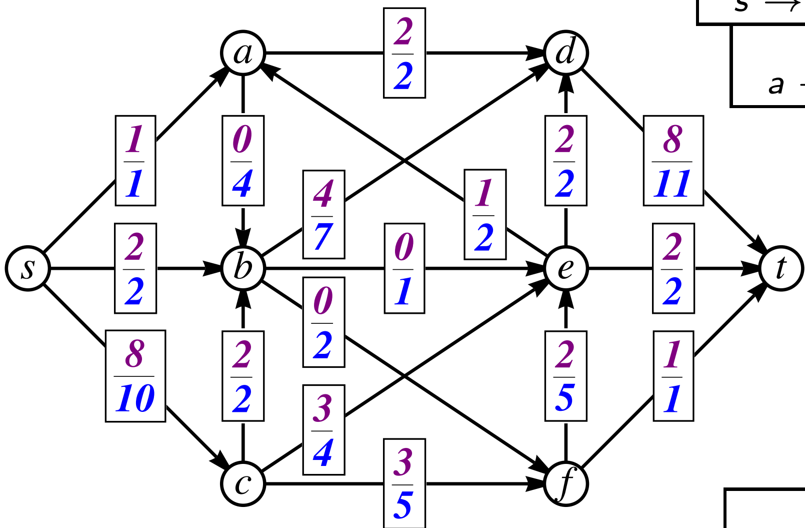


$s \to c \to b \to d \to t$: 2 units
$s \to c \to f \to e \to d \to t$: 2 units



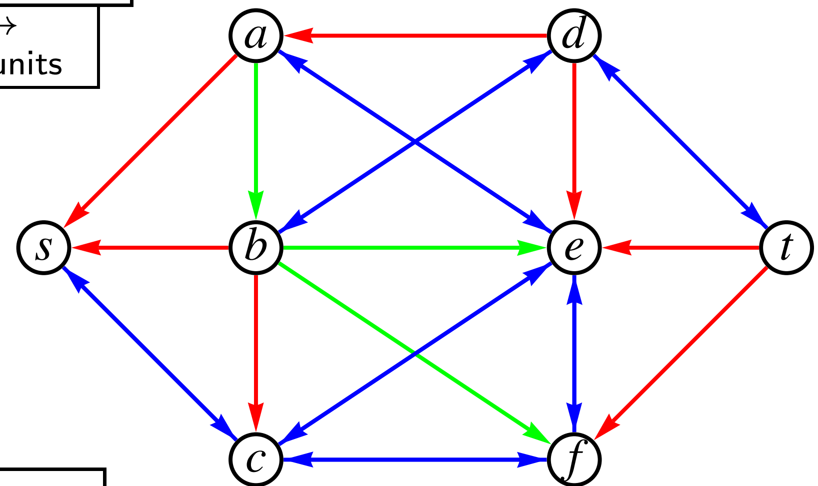


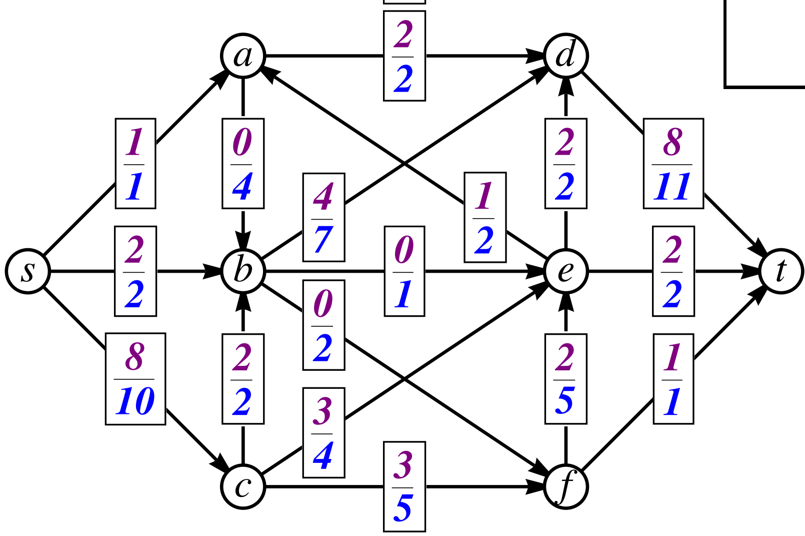
A Ford–Fulkerson Algorithm Example

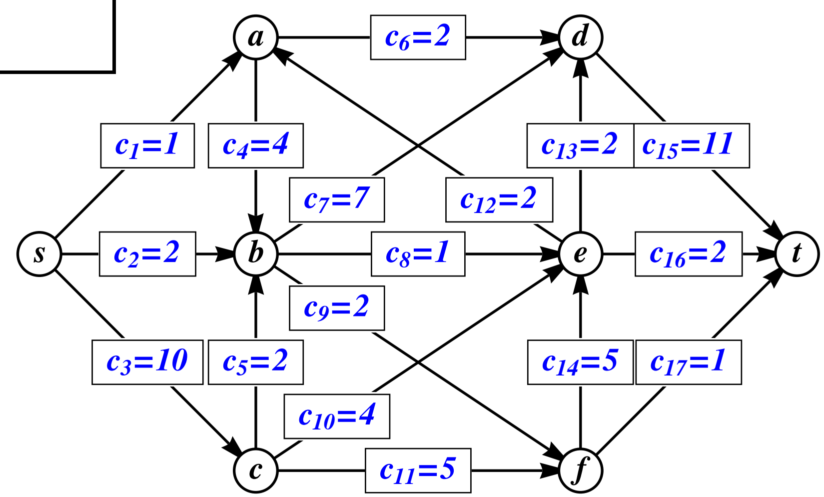


$s \to c \to f \to t$: 1 unit

$s \to c \to e \to a \to d \to t$: 2 units







$X = \{\underline{\hspace{3cm}}\}$, $[X, X^c] = \{\underline{\hspace{4cm}}\}$, and $\left|[X, X^c]\right| = \underline{\hspace{1.5cm}}$.

Correctness of the Ford–Fulkerson Algorithm

Claim. The Ford–Fulkerson Algorithm gives a maximum flow.
Proof. We must show that the algorithm always stops, and that when it stops, the output is indeed a maximum flow.

⋆ We will consider the case of integer capacities.

The algorithm terminates.

- Each iteration increases the throughput of the flow by an integer.
- The sum of the capacities on the edges out of s is finite.

The output is a maximum flow. Upon termination:

- There are no flow augmenting paths in the companion graph, so:
- Edges from X to X^c are full and edges from X^c to X are empty.
- The capacity of $[X, X^c]$ equals the throughput of the flow.

Conclusion. The flow is a max flow and the st-cut is a min cut.

Closing Remarks

- When using the algorithm, it is important to increase the flow by as much as possible at each step.

- When the capacities are integers, we always increase the throughput by integers. The algorithm does work when the capacities are not integers, but the proof is more involved.
- As presented here, this algorithm may be very slow.