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People's Preferences

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1^{st}	Alina	Alina	Dakota
2^{nd}	Casper	Dakota	Casper
3^{rd}	Dakota	Casper	Alina

Pets' Preferences

	Alina	Casper	Dakota
1^{st}	Felicia	Basil	Evan
2^{nd}	Basil	Felicia	Felicia
3^{rd}	Evan	Evan	Basil

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Goal: Create a perfect matching of **stable marriages**.

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If Basil prefers Casper to Alina: ____________________

If Basil prefers Alina to Casper: ____________________

The Gale–Shapley Algorithm

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   - If the pet has ≥ 1 proposal (old or new), it uses its preference list to decides which proposal to accept, rejecting all others.

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$<<$Time for your moment of zen$>>$

Applying the Gale–Shapley Algorithm

Here is a complete set of preferences for 4 people and 4 pets.

People:

Emma
Jae
Tracy
Robot Human

People's Preferences

	Emma	Jae	Tracy	Robot
1st	Parrot	Parrot	Parrot	Sally
2nd	Sally	Casper	Dakota	Dakota
3rd	Casper	Sally	Casper	Parrot
4th	Dakota	Dakota	Sally	Casper

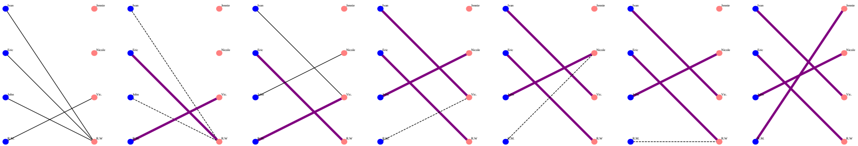
Pets:

Casper the Cat
Dakota the Dog
Sally the Snake
Robot Parrot

Women's Preferences

	Casper	Dakota	Sally	Parrot
1st	Jae	Tracy	Tracy	Jae
2nd	Tracy	Robot	Emma	Robot
3rd	Robot	Jae	Robot	Emma
4th	Emma	Emma	Jae	Tracy

The Algorithm, Pictorially



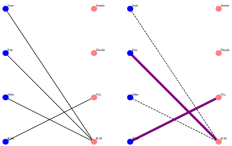
People's Preferences

Emma	Jae	Tracy	Robot
Parrot	Parrot	Parrot	Sally
Sally	Casper	Dakota	Dakota
Casper	Sally	Casper	Parrot
Dakota	Dakota	Sally	Casper

Pets' Preferences

Casper	Dakota	Sally	Parrot
Jae	Tracy	Tracy	Jae
Tracy	Robot	Emma	Robot
Robot	Jae	Robot	Emma
Emma	Emma	Jae	Tracy

The Algorithm, Pictorially



People's Preferences

Emma	Jae	Tracy	Robot
Parrot	**Parrot**	**Parrot**	**Sally**
Sally	Casper	Dakota	Dakota
Casper	Sally	Casper	Parrot
Dakota	Dakota	Sally	Casper

Pets' Preferences

Casper	Dakota	Sally	Parrot
Jae	Tracy	Tracy	**Jae**
Tracy	Robot	Emma	Robot
Robot	Jae	**Robot**	**Emma**
Emma	Emma	Jae	**Tracy**

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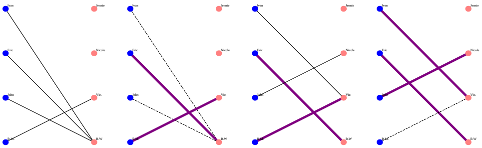

People's Preferences

Emma	Jae	Tracy	Robot
Parrot	**Parrot**	Parrot	**Sally**
Sally	Casper	**Dakota**	Dakota
Casper	Sally	Casper	Parrot
Dakota	Dakota	Sally	Casper

Pets' Preferences

Casper	Dakota	Sally	Parrot
Jae	**Tracy**	Tracy	**Jae**
Tracy	Robot	**Emma**	Robot
Robot	Jae	**Robot**	Emma
Emma	Emma	Jae	Tracy

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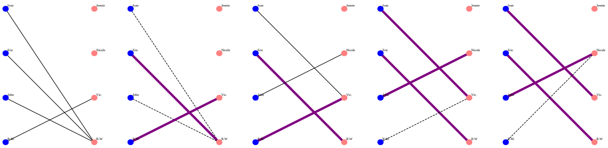

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Parrot	**Parrot**	Parrot	Sally
Sally	Casper	**Dakota**	**Dakota**
Casper	Sally	Casper	Parrot
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Pets' Preferences

Casper	Dakota	Sally	Parrot
Jae	**Tracy**	Tracy	**Jae**
Tracy	**Robot**	**Emma**	Robot
Robot	Jae	Robot	Emma
Emma	Emma	Jae	Tracy

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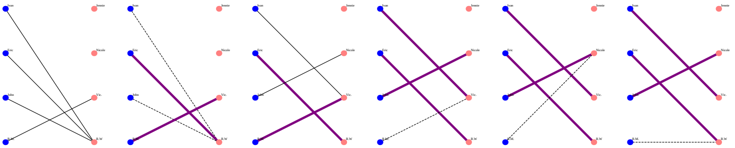

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Casper	Sally	Casper	**Parrot**
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Jae	**Tracy**	Tracy	**Jae**
Tracy	Robot	**Emma**	**Robot**
Robot	Jae	Robot	Emma
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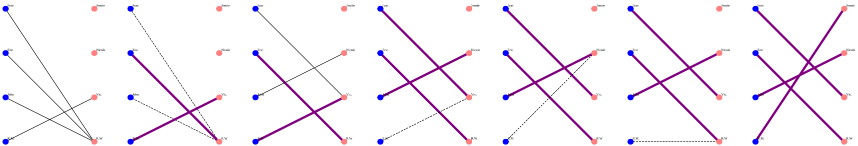

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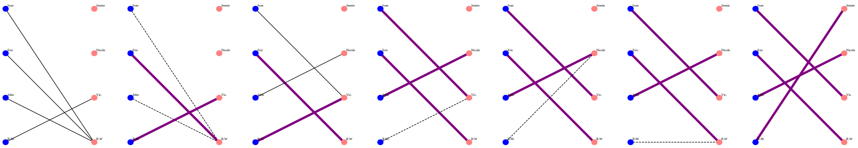

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Claim: Upon termination, everyone is partnered.

- Once a pet finds a partner, it stays partnered.
- If a pet is not partnered at the end, it had no proposal.
- It follows that there is also some person not engaged. However, he/she must have proposed to the lonely pet during some round!

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- Therefore, there is no instability.

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- This, in turn, creates an instability in $\mathcal{S}'$ since P prefers I to H and I prefers P to the pet he is paired with.

***H*-Optimal Pairings ($\mathcal{S}'$)**

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- The National Resident Matching Program implements this algorithm to match medical students to residency programs. (`http://www.nrmp.org`)