
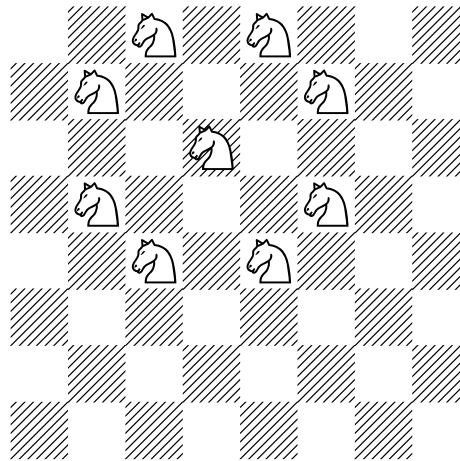


Knight's Tours

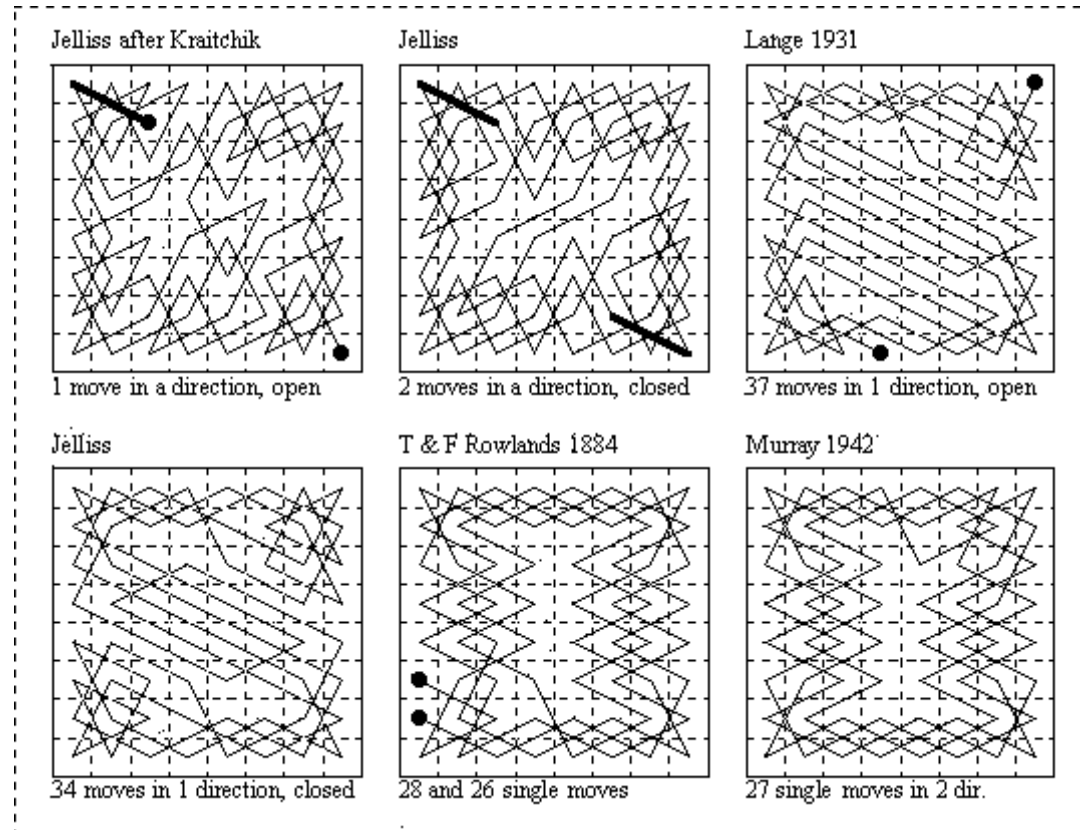
In chess, a knight () is a piece that moves in an “L”: two spaces over and one space to the side.



Question Is it possible for a knight to start on some square and, by a series of valid knight moves, visit each square on an 8×8 chessboard once? (How about return to where it started?)

Definition: A path of the first kind is called an **open knight's tour**. A cycle of the second kind is called a **closed knight's tour**.

8 × 8 Knight's Tour

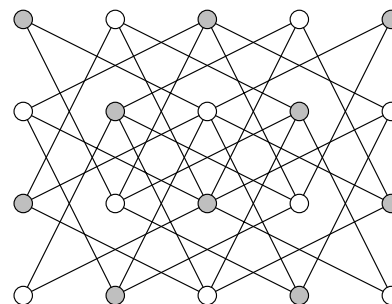
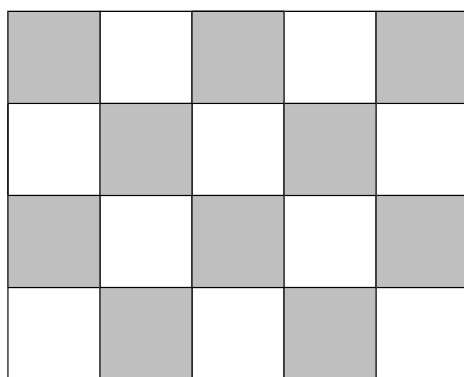


Source: <http://www.ktn.freeuk.com/ga.htm>

Question: Are there any knight's tours on an $m \times n$ chessboard?

The Graph Theory of Knight's Tours

For any board we can draw a corresponding knight move graph: Create a vertex for every square on the board and create edges between vertices that are a knight's move away.



An open/closed knight's tour
on the board

A knight move always alternates between white and black squares.
Therefore, a knight move graph is always _____.

Question Are there any knight's tours on an $m \times n$ chessboard?

Knight's Tour Theorem

Theorem An $m \times n$ chessboard with $m \leq n$ has a *closed* knight's tour unless one or more of these conditions holds:

- 1** m and n are both odd.
- 2** $m = 1, 2,$ or 4 .
- 3** $m = 3$ and $n = 4, 6,$ or 8 .

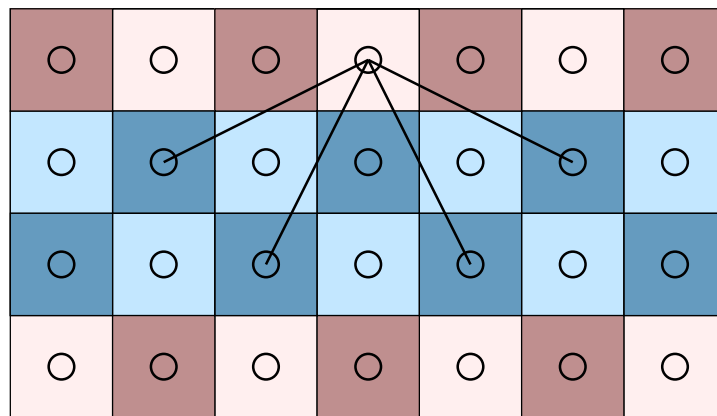
“Proof” We will only show that it is impossible in these cases.

Case 1. When m and n are both odd,

Case 2. When $m = 1$ or 2 , the knight move graph is not connected.

Knight's Tour Theorem

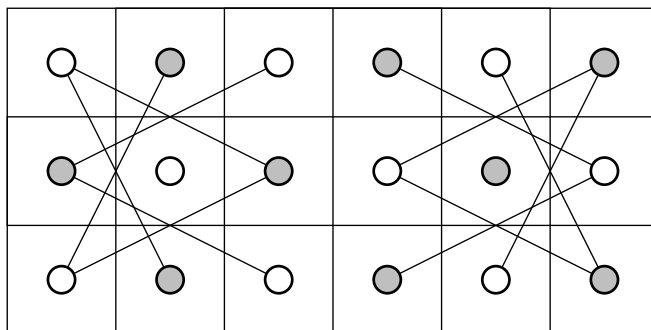
Case 2. When $m = 4$, draw the knight move graph G .



Suppose there exists a Hamiltonian cycle C in the graph G . Since G is bipartite, C alternates between white and black vertices. Notice that every red vertex in C is adjacent to only blue vertices. And, there are the same number of red and blue vertices. So, C must alternate between red and blue vertices. This means: All vertices of C are “white and red” or “black and blue”.

Knight's Tour Theorem

Case 3. 3×4 is covered by Case 2. Consider the 3×6 board:



Assume that there is a Hamiltonian cycle C in G .

Then, C visits each vertex v and uses two of v 's incident edges.

If $\deg(v) = 2$, then both of v 's incident edges are in C .

Draw in all these “forced edges” above. With just these forced edges, there is already a cycle C' of length four. This cycle C' cannot be a subgraph of any Hamiltonian cycle, contradicting its existence. \square

The 3×8 case is similar, and part of your homework.

See also: “Knight's Tours on a Torus”, by J. J. Watkins, R. L. Hoenigman