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Definition: A **decomposition** of a graph G is a set of subgraphs $H_1, \ldots, H_k$ that partition of the edges of G. That is, for all i and j,

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Definition: An H-**decomposition** is a decomposition of G such that each subgraph H_i in the decomposition is isomorphic to H.

Perfect Matching Decomposition

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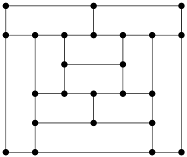
Corollary: K_{2n+1} has a perfect matching decomposition.
Corollary: A snark has no perfect matching decomposition.

Hamiltonian Cycles

Definition: A **Hamiltonian cycle** C in a graph G is a cycle containing every vertex of G.



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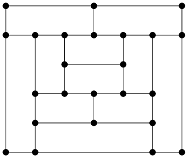

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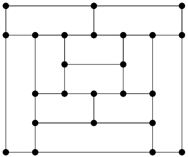

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An arbitrary graph may or may not contain a Hamiltonian cycle/path. This is very hard to determine in general!

⋆ Important: Paths and cycles do not use any vertex or edge twice. ⋆

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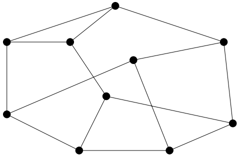
Fact: A snark has an even number of vertices.

Proof: Suppose that a graph G is a snark and contains a Hamiltonian cycle.
That is, G contains C, visiting each vertex once.



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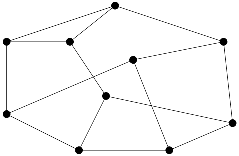

Proof: Suppose that a graph G is a snark and contains a Hamiltonian cycle.
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Remove the edges of C; what remains?

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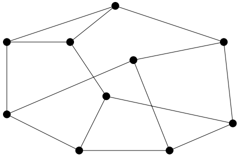

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Consider the coloring of G where the remaining edges are colored yellow and the edges in the cycle are colored alternating between blue and red. This is a proper 3-edge-coloring of G, a contradiction.

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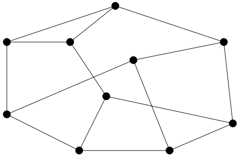

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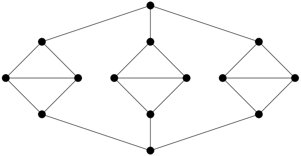
The converse is not true!

Example: Book Figure 2.3.4.



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Definition: A **Hamiltonian cycle decomposition** is a decomposition such that each subgraph H_i is a Hamiltonian cycle.

Question: Which graphs have a Hamiltonian cycle decomposition?

Which complete graphs?

Hamiltonian Cycle Decomposition

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Hamiltonian Cycle Decomposition

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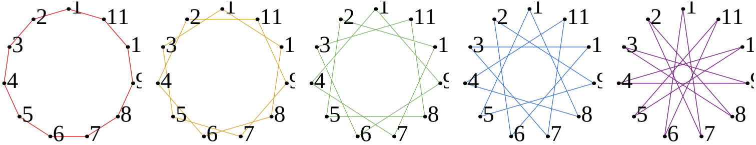
Example: K_{11} has a Hamiltonian cycle decomposition.



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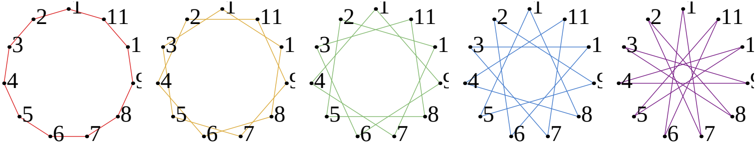
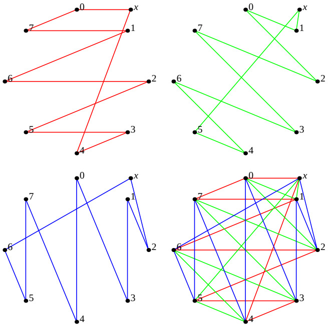


However: This construction does not work with K_9.

Hamiltonian Cycle Decomposition

Theorem 2.3.1: K_{2n+1} has a Hamiltonian cycle decomposition.

Proof: This proof uses another instance of a "turning trick". Place vertices 0 through $2n$ in a circle and draw a zigzag path visiting all the vertices in the circle.



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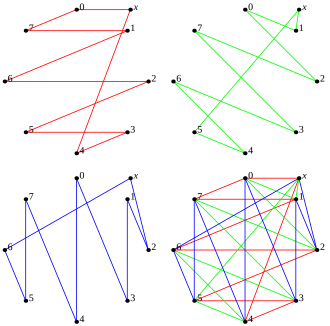

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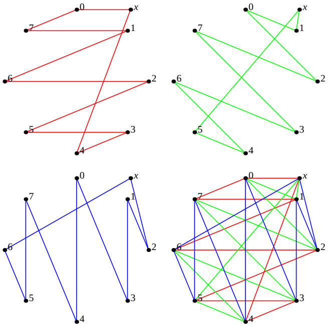

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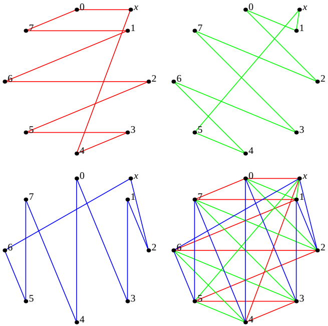

As a corollary:

Theorem 2.3.3: K_{2n} has a Hamiltonian path decomposition.