# Special subgraphs

*Definition:* (pp. 20,34) Let $G$ be a graph.
A subgraph $H$ is **spanning** if $H$ contains all of $G$'s vertices.

- ▶ If $H$ is spanning and also a tree, $H$ is called a **spanning tree**.
- ▶ If $H$ is spanning and also $r$-regular, then $H$ is called an $r$-**factor**.
  A 1-factor is also called a **perfect matching**.
  For a graph $G$ to have a perfect matching, _____.

*Definition:* A **decomposition** of a graph $G$ is a set of subgraphs
$H_1, \ldots, H_k$ that partition of the edges of $G$. That is, for all $i$ and $j$,

$$\bigcup_{1 \leq i \leq k} H_i = G \text{ and } E(H_i) \cap E(H_j) = \emptyset.$$

*Definition:* An $H$-**decomposition** is a decomposition of $G$ such
that each subgraph $H_i$ in the decomposition is isomorphic to $H$.

# Perfect Matching Decomposition

*Definition:*  A **perfect matching decomposition** is a decomposition such that each subgraph $H_i$ in the decomposition is a perfect matching.

*Theorem:*    For a $k$-regular graph $G$,
$G$ has a perfect matching decomposition if and only if $\chi'(G) = k$.

*Proof:*
  There exists a decomposition of $G$ into a set of $k$ perfect matchings.
$$\Updownarrow$$
    There exists a coloring of the edges of $G$ where each vertex is incident to edges of each of $k$ different colors.
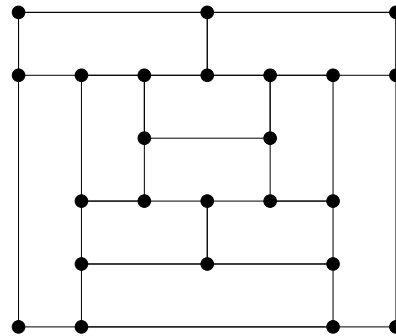$$\Updownarrow$$
$$\chi'(G) = k$$

*Corollary:*    $K_{2n+1}$ has a perfect matching decomposition.
*Corollary:*    A snark has no perfect matching decomposition.

# Hamiltonian Cycles

*Definition:* A **Hamiltonian cycle** $C$ in a graph $G$ is a cycle containing every vertex of $G$.



*Definition:* A **Hamiltonian path** $P$ in a graph $G$ is a path containing every vertex of $G$.

*Theorem:* If $G$ has a Ham'n cycle, then $G$ has a Ham'n path.
*Proof:*

An arbitrary graph may or may not contain a Hamiltonian cycle/path. This is very hard to determine in general!

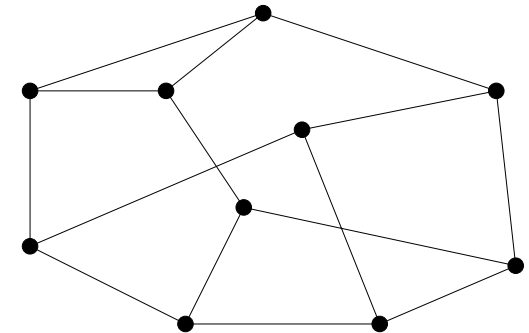⋆ Important: Paths and cycles do not use any vertex or edge twice. ⋆

# Hamiltonian Cycles

*Theorem 2.3.5:*    A snark has no Hamiltonian cycle.

*Fact:*    A snark has an even number of vertices.

*Proof:*    Suppose that a graph $G$ is a snark and contains a Hamiltonian cycle.
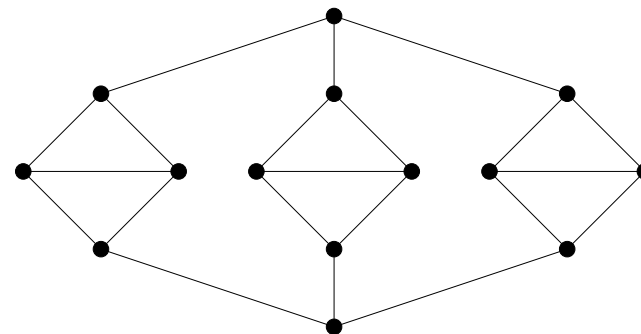That is, $G$ contains $C$, visiting each vertex once.
Remove the edges of $C$; what remains?

Consider the coloring of $G$ where the remaining edges are colored yellow and the edges in the cycle are colored alternating between blue and red. This is a proper 3-edge-coloring of $G$, a contradiction.

*The converse is not true!*

*Example:* Book Figure 2.3.4.

# Cycle Decompositions

*Definition:*  A **cycle decomposition** is a decomposition such that each subgraph $H_i$ in the decomposition is a cycle.

*Theorem:*   Let $G$ be a graph that has a cycle decomposition. Then every vertex of $G$ has even degree.

*Proof:*   In the cycle decomposition, each cycle $H_i$ contributes two to the degree of each of its vertices. For a vertex $v$ in $G$, its degree is the sum of its degree over all subgraphs $H_i$, which must be even.
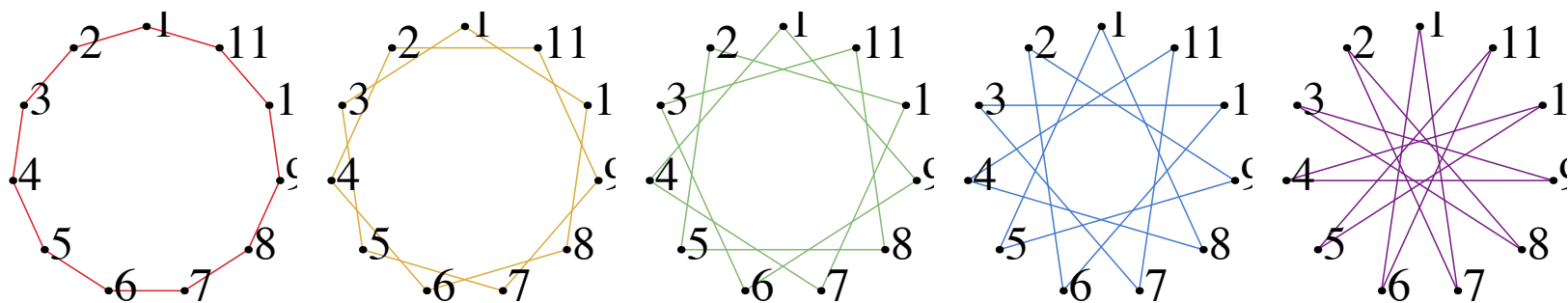
*Definition:*  A **Hamiltonian cycle decomposition** is a decomposition such that each subgraph $H_i$ is a Hamiltonian cycle.

*Question:*   Which graphs have a Hamiltonian cycle decomposition?

Which complete graphs?

# Hamiltonian Cycle Decomposition

*Example:*   $K_7$ has a Hamiltonian cycle decomposition.

*Example:*   $K_{11}$ has a Hamiltonian cycle decomposition.



*However:*   This construction does not work with $K_9$.

# Hamiltonian Cycle Decomposition

*Theorem 2.3.1:*   $K_{2n+1}$ has a Hamiltonian cycle decomposition.

*Proof:*   This proof uses another instance of a "turning trick". Place vertices 0 through $2n$ in a circle and draw a zigzag path visiting all the vertices in the circle.   Connect the ends of the path to vertex $x$ to form a Ham. cycle.   As you rotate the zigzag path $n$ times, you visit each edge of $K_{2n+1}$ once to form a Ham'n cycle decomposition.
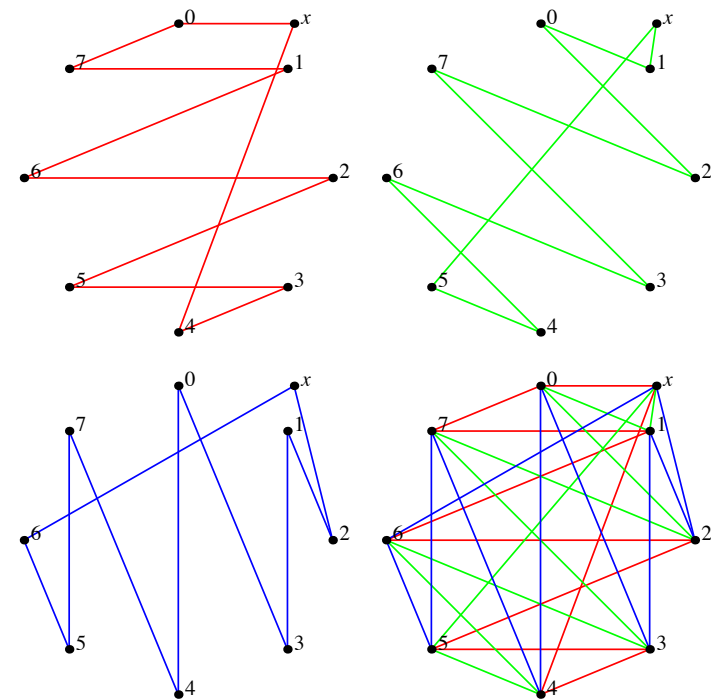


As a corollary:

*Theorem 2.3.3:*   $K_{2n}$ has a Hamiltonian path decomposition.