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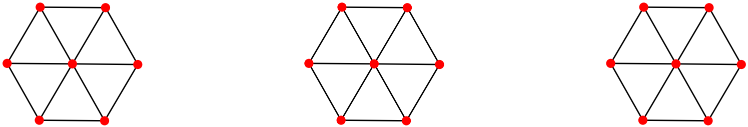

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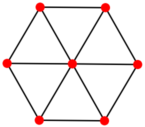

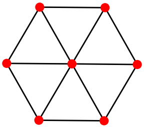

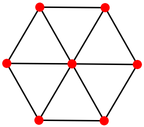

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Of interest: What is the fewest colors necessary to properly color G?

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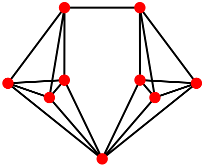
Example. Calculate $\chi(G)$ for this graph G:



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Since G is finite, there will be some proper subgraph G_l of G_{l-1} such that G_l is critical and $\chi(G_l) = \chi(G_{l-1}) = \cdots = \chi(G)$.

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Similarly: If G is critical, then for all $v \in V(G)$, $\deg(v) \geq \chi(G) - 1$.

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$(\Rightarrow)$ Let G be bipartite. Assume that there is some cycle C of odd length contained in $G \ldots$

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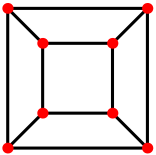

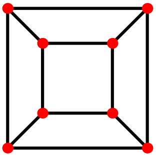

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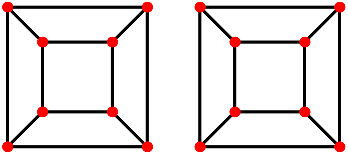

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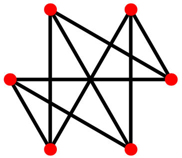
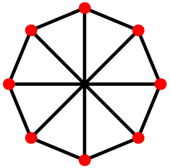
Fact: **Most** 3-regular graphs have edge chromatic number 3.





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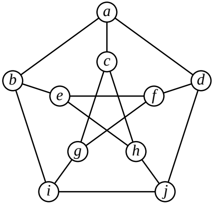

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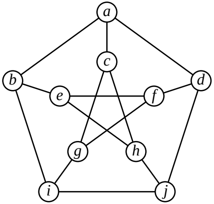

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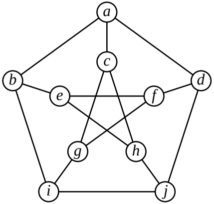

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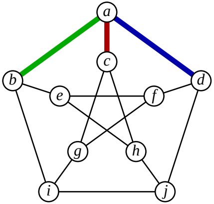

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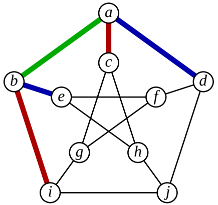

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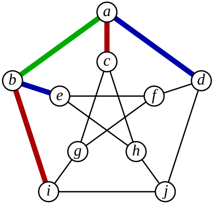

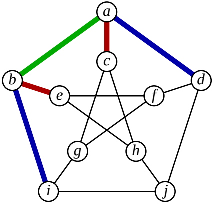

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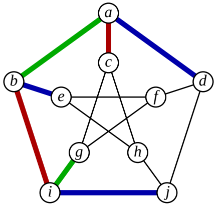

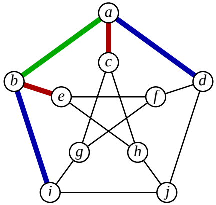

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Definition: A **snark** is a cubic graph with edge chromatic number 4.

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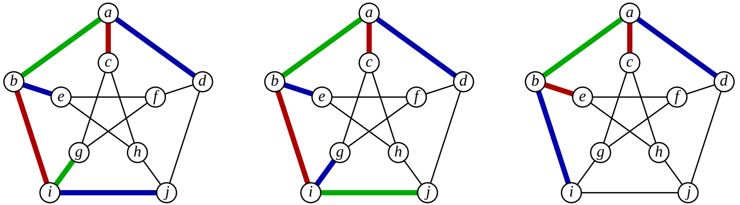

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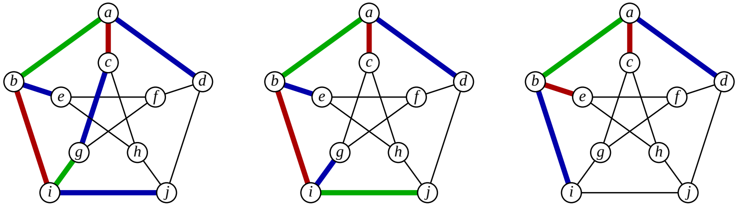

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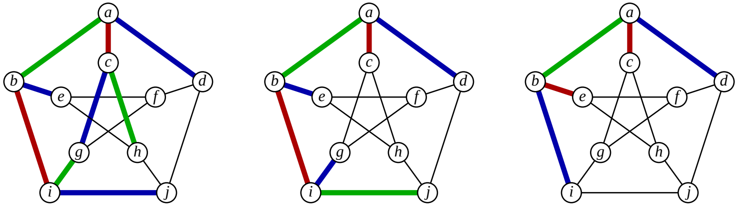

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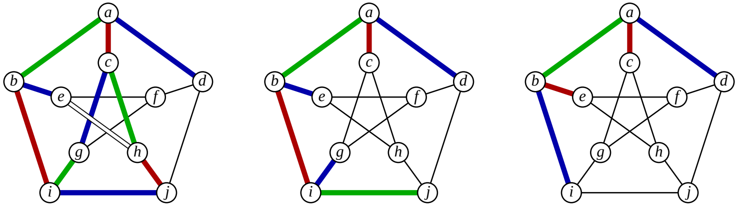

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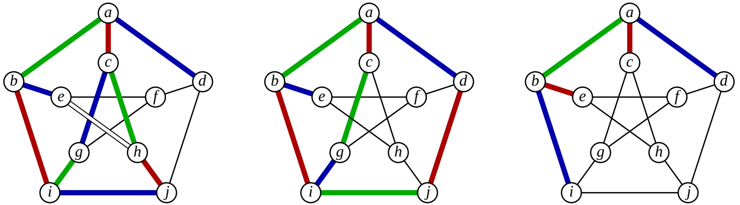

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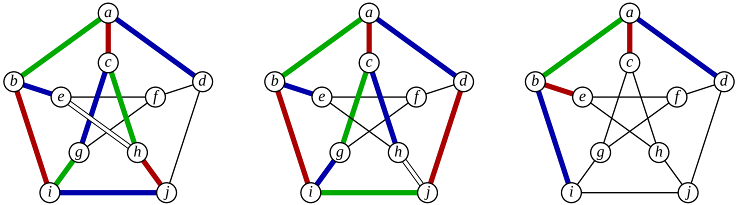

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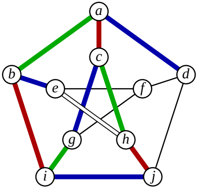

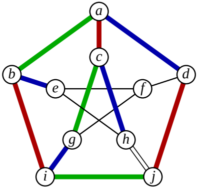

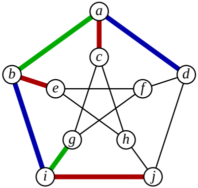

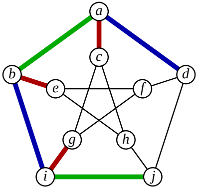

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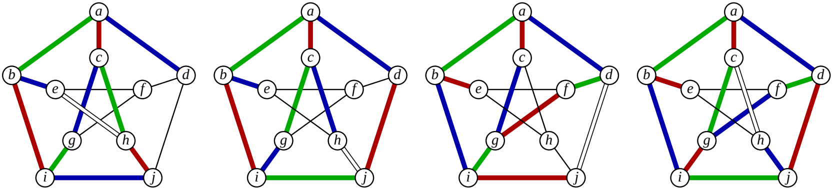

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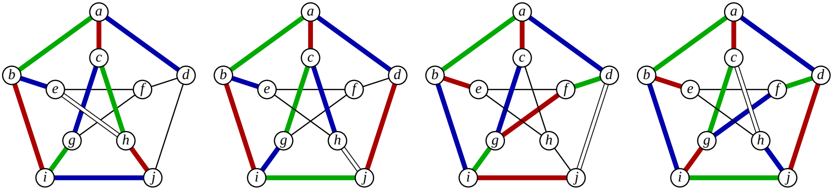

In all cases, it is not possible to edge color with 3 colors, so $\chi'(G) = 4$.

The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

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This is only an integer when:

So, the best we can expect is that $\begin{cases} \chi'(K_{2n}) = \\ \chi'(K_{2n-1}) = \end{cases}$

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Connect 0 with x,

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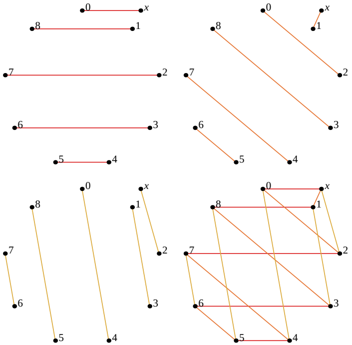

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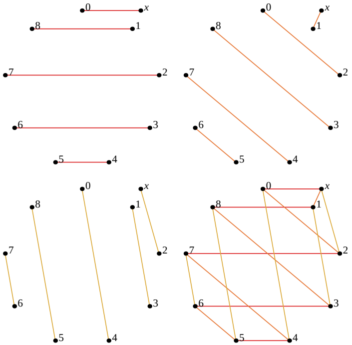

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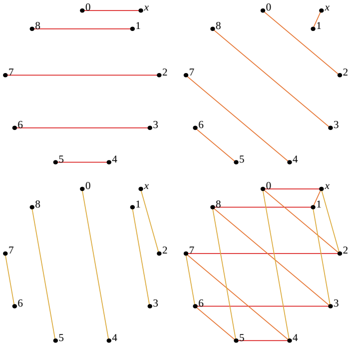

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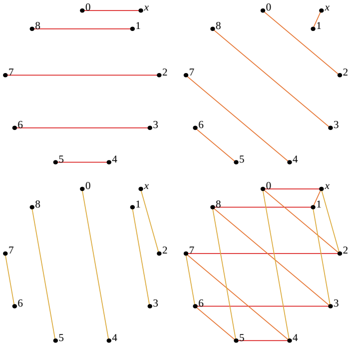

Each time, new edges are used. This is because each of the edges is a different "circular length": vertices are at circ. distance 1, 3, 5, ..., 4, 2 from each other, and x is connected to a different vertex each time.

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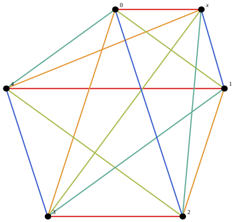
Day 1	0x	14	23
Day 2	1x	20	34
Day 3	2x	31	40
Day 4	3x	42	01
Day 5	4x	03	12



Theorem 2.2.3 proves there is such a tournament for all even numbers.