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Proof. By hypothesis,

- There exist paths $P : av_1v_2 \cdots v_kb$ and $Q : bw_1w_2 \cdots w_lc$ in G.
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- If not all vertices are distinct, then choose the *first* vertex v_p in P that is also a vertex w_q in Q.

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Apply Lemma A to show there is a path from v to w in H.

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Break into cases, depending on whether G contains a cycle:

(next page)

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$\star$ Important Induction Item: Always **remove** edges. $\star$

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Definition: A **tree** is a connected graph that contains no cycles.
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Trees are the smallest connected graphs; the following theorems show this and help classify graphs which are trees.

Thm 1.3.2, 1.3.3: Let G be a connected graph with p vertices and q edges. Then, G is a tree $\Longleftrightarrow p = q + 1$.

Thm 1.3.5. G is a tree iff there exists exactly one path between each pair of vertices.

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Therefore $p \neq q + 1$.

Proof of Theorem 1.3.5

Thm 1.3.5. G is a tree iff there exists exactly one path between each pair of vertices.

($\Rightarrow$) Suppose that G is a tree. Then G is connected, so for all $v_1, v_2 \in V$, there exists at least one path between v_1 and v_2. Suppose that there are two paths, $P_1 = v_1 u_1 u_2 \cdots u_n v_2$ and $P_2 = v_1 w_1 w_2 \cdots w_m v_2$.

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In both cases, it is not the case that between each pair of vertices, there exists exactly one path.

Related theorems

Definition: A **bridge** is an edge e such that its removal disconnects G.

Theorem 2.4.1. Suppose that G is a connected. Then

$$G \text{ is a tree} \iff \text{Every edge of } G \text{ is a bridge.}$$

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Proof. Let G be a regular graph of even degree with a bridge $e = vw$.